# An Algebraic Analysis Approach to Trajectory Tracking Control( Dissertation_全文 )

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# An Algebraic Analysis Approach to Trajectory Tracking Control

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# Abstract

The main aim of this thesis is to clarify a class of nonlinear systems described by ordinary differential equations and reference trajectories such that trajectory tracking controls are easily realized. To this end, the thesis shows a class of nonlinear systems whose linearizations are uniformly completely controllable and uniformly completely observable. On this account, the thesis introduces two novel concepts called algebraic controllability and algebraic observability. In order to characterize them, this thesis also introduces new concepts called controllable trajectory and observable trajectory. It is indicated that if a given nonlinear system is differentially flat, controllable and observable trajectory can be easily generated. As a main result, it is shown that if a given nonlinear system is algebraically controllable, then every linearized system along any (periodic) controllable trajectory is (uniformly) completely controllable. As a dual result, it is also shown that if a given nonlinear system is algebraically observable, then every linearized system along any (periodic) observable trajectory is (uniformly) completely observable. Moreover, it is explained that if a given system is algebraically controllable and observable, a linear quadratic optimal control method is useful to design a feedback controller such that the actual trajectory asymptotically approaches a periodic reference trajectory. Furthermore, the thesis proves that the concepts of algebraic controllability and accessibility are equivalent and for nonlinear mechanical control systems, a reduction condition for checking whether or not the given system is algebraically controllable is provided.

The contributions of the second in the thesis is to give a class of nonlinear differential algebraic systems (DAS) with geometric index one and reference trajectories such that trajectory tracking controls are easily realized. Furthermore, it is demonstrated that it is difficult to examine differential flatness in the usual sense of nonlinear systems expressed by DAS with geometric index one. To resolve the problem, the thesis provides an extended definition of differential flatness for such systems. Moreover, for general DAS, it is also explained that a choice of independent variables is not obvious because there are algebraic equations. For this reason, the thesis studies differential flatness for DAS which does not distinguish state, input, and output variables. As a result, it is shown that if one could find a flat output, one can find other flat outputs by smooth functions of the flat output.

The chapter 4 elaborates the differences between variational and flatness-based trajectory generation methods. Moreover, using a nonholonomic mobile robot described by ordinary differential equations and a simple circuit model expressed by DAS with geometric index one, it is demonstrated that trajectory tracking controls of algebraically controllable and observable systems are easily achieved.

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# Notations

The following notation is used in this thesis.

| | |
|---|---|
| $\boldsymbol{Z}$ | set of integer numbers |
| $\boldsymbol{R}$ | real number field |
| $\boldsymbol{R}_+$ | set of nonnegative real numbers |
| $C^\infty(X, Y)$ | set of smooth functions from $X$ to $Y$ |
| $C^\infty_{\text{a.e.}}$ | set of smooth functions except for a countable set |
| $C^\infty_{\text{pw}}$ | subset of $C^\infty_{\text{a.e.}}$ whose elements are piecewise smooth functions defined on $\boldsymbol{R}$ |
| $\in$ | belong to |
| $\subset$ | subset (strict or not) |
| $0_{n\times m}$ (or simply 0) | $n \times m$ zero matrix |
| $I_n$ (or simply $I$) | $n \times n$ identity matrix |
| $M^\top$ | transpose of the matrix $M$ |
| $M^{-1}$ | inverse of the matrix $M$ |
| $\text{rank}(M)$ | rank of the matrix $M$ |
| $\text{diag}(a_1, a_2, \ldots, a_m)$ | $m \times m$ diagonal matrix with $a_i$ as its $i$-th diagonal element |
| $\|x\|$ | the Euclidean norm of the vector $x \in \mathbb{R}^n$ |
| $\|\|A\|\|$ | $\|\|A\|\| := \max_{x\neq 0} \frac{\|\|Ax\|\|}{\|\|x\|\|}$ for $A \in \boldsymbol{R}^{n\times n}$ |

# Chapter 1

# Introduction

Autonomous systems have received increased high attention because for mechanical systems the needs arise to perform tasks in many situations. Many control problems of such systems consist of following a desired trajectory. In fact, there have been many previous works on trajectory tracking control problems [5, 18, 19, 22, 33, 39, 44–46, 51–53, 61, 62, 67, 68, 72, 73, 76, 79–81, 97, 99, 105, 108, 109], and the following applications can be considered.

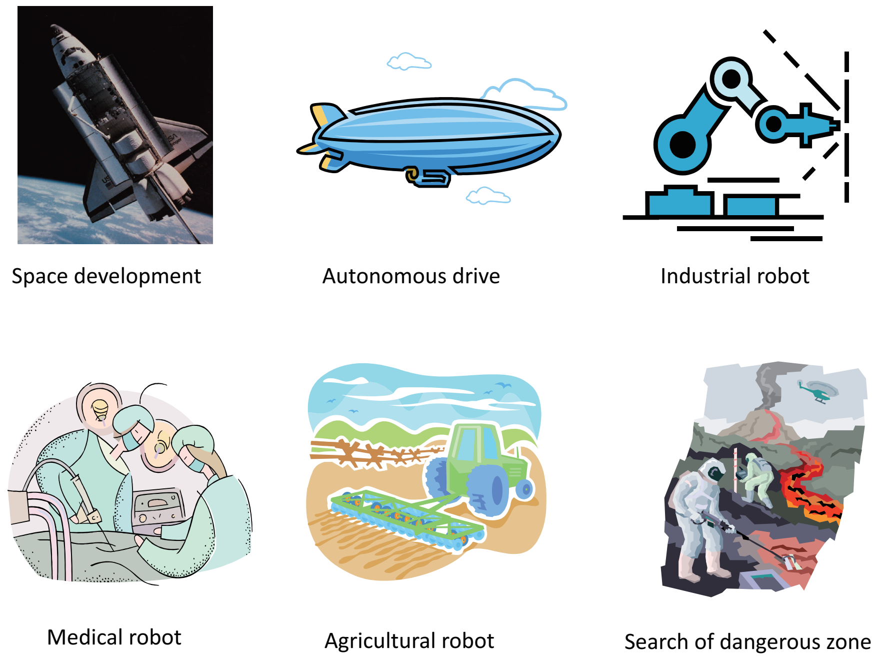



Figure 1.1: Applications of trajectory tracking control

**The main aim of this thesis is to give a class of nonlinear systems and reference trajectories such that trajectory tracking controls are easily achieved.**

## 1.1 Trajectory tracking control

To achieve a trajectory tracking control of a given nonlinear system, the given system is often transformed into a simple system by applying a nonlinear coordinate transformation and a nonlinear feedback [36, 76, 78, 79, 82]. In particular, it is known that exact linearization method for affine nonlinear systems is useful to design a stabilizing controller [36, 82] (see appendix F). A trajectory tracking control based on exact linearization method can be regarded as a **two-degree-of-freedom control** [108] (see appendix F). Two-degree-of-freedom controller design technique is composed of the following procedures (see Figs. 1.2 and 1.3).

1. Give a reference trajectory of a given system.
2. Apply an appropriate feedforward control input such that the actual trajectory approximates the reference trajectory.
3. In order to stabilize the actual trajectory around the reference trajectory, use an appropriate state feedback control together with the feedforward control.

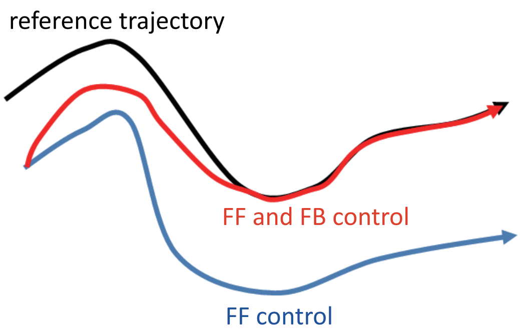



Figure 1.2: Trajectory tracking control

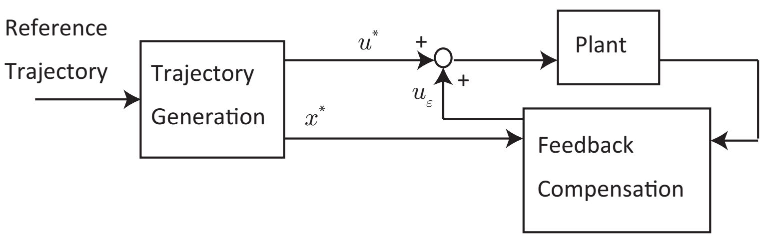



Figure 1.3: Two-degree-of-freedom controller design

However, in general, it is difficult to apply exact linearization method because many practical cases are difficult to obtain an appropriate coordinate transformation. Moreover, if one can get an appropriate coordinate transformation, and if the transformation is applied, the state constraint may be produced because such a coordinate transformation is only locally defined [36, 82]. Furthermore, when there is an input saturation requirement, a feedback based on exact linearization method violates the input saturation requirement [39] (see appendix F.2). To avoid the difficulties, the thesis uses linear approximation along a trajectory, instead of using exact linearization method.

At the above step 3 of two-degree-of-freedom control, the resulting error system between the reference trajectory and the actual trajectory is approximately expressed as a linear time varying system if the actual trajectory is sufficiently close to the reference trajectory. If a linear feedback controller of the linearized system such that the origin is exponentially stabilizable is designed, by applying the same controller into the original nonlinear error system, the origin of the closed-loop of the original nonlinear error system is locally exponentially stable [48]. Consequently, then the actual trajectory locally exponentially approaches the reference trajectory. Uniform complete controllability of the linearized error system is a sufficient condition for the origin of the linearized error system to be exponentially stabilizable [34]. Although it is possible to examine whether or not uniform complete controllability is satisfied by using conventional methods, the examination can be carried out **only** for a **fixed** linearized system along a **specific** trajectory. That is, it is not clear what is a class of nonlinear systems whose linearizations are uniformly completely controllable. Therefore the following questions are posed.

**Question 1.1** *What is a class of nonlinear systems whose linearizations along trajectories are uniformly completely controllable?*

**Question 1.2** *What is a class of trajectories stated in the above question?*

In chapter 2, it will be shown that if a given nonlinear system described by ordinary differential equations is algebraically controllable, then every linearized system along any (periodic) controllable trajectory is (uniformly) completely controllable. In chapter 3, it will be shown that similar results are obtained for nonlinear differential algebraic systems with geometric index one.

On the other hand, if the available signal in nonlinear systems is only output signal, state feedback cannot be available. In this case, to stabilize the actual trajectory around the reference trajectory, it is needed to design a state observer such as a Luenberger type observer. It is known that if the linearized error system is uniformly completely controllable and uniformly completely observable, then there exist a feedback gain and an observer gain such that the origin of a linear closed-loop obtained by applying a state-estimate feedback is exponentially stable [35]. Although it is possible to examine whether or not uniform complete observability is satisfied by using conventional methods, the examination can be carried out **only** for a **fixed** linearized system along a **specific** trajectory. That is, it is not clear what is a class of nonlinear systems whose linearizations are uniformly completely observable. Therefore the following questions are also posed.

**Question 1.3** *What is a class of nonlinear systems whose linearizations along trajectories are uniformly completely observable?*

**Question 1.4** *What is a class of feasible trajectories stated in the above question?*

In chapter 2, it will be shown that if a given nonlinear system described by ordinary differential equations is algebraically observable, then every linearized system along any (periodic) observable trajectory is (uniformly) completely observable. In chapter 3, it will be shown that similar results are obtained for nonlinear differential algebraic systems with geometric index one.

## 1.2 Algebraic approach in systems theory

In order to answer the questions 1.1, 1.2, 1.3, 1.4, this thesis applies algebraic approach in systems theory. Behavioral theory [88, 111–113] for linear systems has applied algebraic approach [14,83,90,115,116,118,119]. Concretely, algebraic analysis [47] has applied to study controllability and observability properties of a behavior defined by solutions of linear ordinary (partial) differential equations [14,83,90,115,116,118,119]. In particular, when a behavior is defined by solutions of linear ordinary differential equations with meromorphic coefficients, it has been shown that the Jacobson form of a polynomial matrix, whose each element is composed of a differential operator with meromorphic coefficients, is useful for an algebraic analysis of system's properties [118].

On the other hand, for nonlinear systems, differential algebra was introduced by J. F. Ritt [96] as an extension of commutative algebra for algebraic equations. In particular, it was introduced to study differential algebraic equations and was first applied to nonlinear control systems by M. Fliess to resolve the problem of invertibility of nonlinear input-output differential systems in [20]. Currently, there exist two approaches on differential algebraic analysis in nonlinear control theory. The first approach frequently uses Kähler differentials introduced in [40]. This is purely algebraic. For example, see [17,21,22,29] and references therein. On the other hand, the second approach uses pseudo-linear algebra [8] (see appendix C). This approach applies the formal vector space of differential one-forms, which is not purely algebraic. For example, see [2, 16, 28, 55, 56, 120] and references therein. G. Fu etc. [24] have shown that a quotient space of Kähler differentials is isomorphic to the formal vector space of differential one-forms. These two spaces coincide if they are over the field of algebraic functions. The thesis adopts the latter approach, that is, this thesis applies pseudo-linear algebra to nonlinear systems.

In order to examine whether or not a given nonlinear system is algebraically controllable (observable), it is possible to use computer algebra such as Mathematica and Maple. In particular, since algebraic controllability (observability) is defined by using the Jacobson form of a skew polynomial matrix derived from a given nonlinear system, they can be checked by applying computer algebra based on Gröbner basis theory [63, 64, 74, 100].

## 1.3 Trajectory generation

In order to achieve trajectory tracking control by using two-degree-of-freedom controller design techniques, given a reference trajectory, it is required to apply an appropriate feedforward control. That is, it is needed to carry out the step 2 as mentioned in section 1.1. To this end, it is possible to apply optimal control methods such as a variational method and a dynamic programming method [103]. For a nonlinear system, to solve an optimal control problem by using a variational method, it is needed to solve a nonlinear ordinary differential equation called an Euler-Lagrange equation. On the other hand, to solve an optimal control problem by using a dynamic programming method, it is required to solve a nonlinear partial differential equation called a Hamilton-Jacobi-Bellman (HJB) equation. Hence, if a given system is nonlinear, it is difficult to solve optimal control problems. Fortunately, it has been known that if a given nonlinear system is differentially flat, trajectory generations of the system are very easy [5, 19, 22, 52, 53, 67, 68, 72, 73, 79–81, 97, 99, 105, 108, 109]. M. Fliess etc. [22] first introduced the concept of differential flatness in a differential algebraic context [54, 96] and later in a differential geometric context by using Lie-Bäcklund transformation [23] (see appendix E). State and control input values of differentially flat systems are completely determined by a set of variables called a flat output composed of as many variables as input variables [22, 23, 65]. On this account, a flatness-based trajectory generation method is easy compared with optimization techniques by solving an Euler-Lagrange equation or a HJB equation. In fact, given initial and final states, they possess the following characteristics for generating trajectories connecting their states.

- Variational method: Solve an Euler-Lagrange equation (nonlinear ordinary differential equation). Then a feedforward control input is obtained.
- Dynamic programming method: Solve a HJB equation (nonlinear partial differential equation). Then a feedback control input is obtained.
- Flatness-based trajectory generation method: Solve **linear algebraic** equations. Then a feedforward control input is obtained.

Chapter 4 elaborates the differences between variational and flatness-based trajectory generation methods. Note that although a flatness-based trajectory generation method in chapter 4 does not guarantee to generate an optimal trajectory, there are some works on a flatness-approach which guarantees optimality [19, 67, 68, 98]. Reference [98] has pointed out that a flatness-approach frequently converts the original convex constraints to non-convex constraints. To resolve the problem, references [67,68] have studied convex approximations of the non-convex constraints inspired by [19].

In general, given initial and final states, there are infinite trajectories which concatenates their states (See Fig. 1.4).

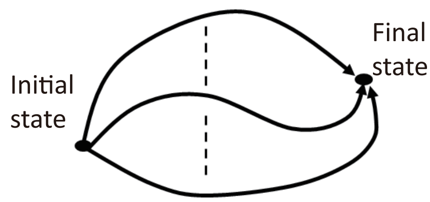



Figure 1.4: Trajectories connecting initial and final states

Thus it is important to choose a "good" trajectory from among the infinite trajectories. For this reason, the thesis presents the following statements.

- Suppose that a given nonlinear system is **algebraically controllable** and a periodic **controllable trajectory** is given. Let the reference trajectory be the state part of the controllable trajectory. Then it is possible to design a controller such that the actual trajectory locally exponentially approaches the reference trajectory.
- Suppose that a given nonlinear system is **algebraically observable** and a periodic **observable trajectory** is given. Let the reference trajectory be the state part of the observable trajectory. Then it is possible to design an observer gain such that the origin of error dynamics between the actual error state and the estimated error state of the linearized system of the given nonlinear system is exponentially stable. Moreover, if the nonlinear system is also algebraically controllable and the observable trajectory is also controllable trajectory, it is expected that one can design a controller and an observer such that the actual trajectory locally exponentially approaches the reference trajectory.

## 1.4 Contributions and organization of the thesis

The main aim of this thesis is to clarify a class of nonlinear systems such that trajectory tracking controls are easily realized. To this end, the thesis gives a class of nonlinear systems whose linearizations along certain trajectories are uniformly completely controllable and uniformly completely observable. In order to describe such a class, algebraic controllability and algebraic observability are introduced. Moreover, to characterize them, the concepts of controllable trajectory and observable trajectory are also introduced.

The contributions of the thesis are as follows:

- Novel concepts such as algebraic controllability, algebraic observability, controllable trajectory, and observable trajectory are introduced (Chapter 2).
- It is shown that if a given nonlinear system described by ordinary differential equations is algebraically controllable, every linearized system along any (periodic) controllable trajectory is (uniformly) completely controllable. As a dual result, it is shown that if a given nonlinear system described by ordinary differential equations is algebraically observable, every linearized system along any (periodic) observable trajectory is (uniformly) completely observable (Chapter 2).
- It is shown that the concepts of algebraic controllability and accessibility are equivalent (Chapter 2).
- For nonlinear mechanical control systems, a reduction condition for checking whether or not the system is algebraically controllable is provided (Chapter 2).
- For nonlinear differential algebraic systems (DAS) with geometric index one, algebraic controllability, algebraic observability, controllable trajectory, and observable trajectory are also introduced. Furthermore, it is shown that similar results to ordinary differential equations are obtained (Chapter 3).
- It is given the definition of differential flatness of DAS and provided how to produce other flat outputs from a given flat output (Chapter 3).
- It is clarified the difference between variational and flatness-based trajectory generation methods (Chapter 4).
- It is demonstrated that LQ optimal control and LMI methods are useful to design for a trajectory tracking control of algebraically controllable and observable systems (Chapter 4).

This thesis is organized as follows.

Chapter 2 first explains in detail that the concepts of uniform complete controllability and uniform complete observability are useful for trajectory tracking control. Next, the concepts of algebraic controllability and algebraic observability are introduced and to characterize them, controllable trajectory and observable trajectory are also introduced. As a main result, it is shown that if a given nonlinear system is algebraically controllable, every linearized system along any (periodic) controllable trajectory is (uniformly) completely controllable. As a dual result, it is shown that if a given nonlinear system is algebraically observable, every linearized system along any (periodic) observable trajectory is (uniformly) completely observable. It is explained that although the concepts of controllable trajectory and observable trajectory is not required in the case of linear systems, they are needed in the case of nonlinear systems. Moreover, it is explained that if a given system is algebraically controllable and observable, a linear quadratic optimal control method is useful to design a feedback controller such that the actual trajectory asymptotically approaches a periodic reference trajectory. Furthermore it is proven that the concept of algebraic controllability coincides with the concept of accessibility. Finally, for nonlinear mechanical control systems, a reduction condition for algebraic controllability is provided.

Chapter 3 introduces the concepts of algebraic controllability and algebraic observability for nonlinear DAS with geometric index one and shows the results similar to main results of chapter 2. Furthermore, it is explained that it is difficult to examine differential flatness in the usual sense of nonlinear systems described by DAS with geometric index one, and, as a result, it makes difficult to generate controllable and observable trajectory. To resolve this problem, the definition of differential flatness is extended for DAS with geometric index one. Moreover, for general DAS, a choice of independent input variables is not obvious because there are algebraic equations. Hence it is meaningful to study DAS which does not distinguish state, input, and output variables. For this reason, the definition of differential flatness of general DAS and how to produce other flat outputs from a given flat output are provided.

Chapter 4, first, elaborates the difference between variational and flatness-based trajectory generation methods. Concretely, using a nonholonomic mobile robot model, it is shown that although in the case of a variational method, it is required to solve nonlinear differential equations, it is just needed to solve linear algebraic equations in the case of a flatness-based trajectory generation. Moreover, using a nonholonomic mobile robot expressed by ordinary differential equations and a simple circuit model expressed by DAS with geometric index one, it is demonstrated that trajectory tracking controls of algebraically controllable and algebraically observable systems are easily realized.

Chapter 5 concludes the thesis.

# Chapter 2

# Algebraic controllability and algebraic observability

The aim of this chapter is to give a class of nonlinear systems and reference trajectories such that trajectory tracking controls are easily realized. To this end, the chapter shows a class of nonlinear systems whose linearizations are uniformly completely controllable and uniformly completely observable. Concretely, it is shown that if a given nonlinear system is algebraically controllable, then every linearized system along any (periodic) controllable trajectory is (uniformly) completely controllable. As a dual result, it is shown that if a given nonlinear system is algebraically observable, then every linearized system along any (periodic) observable trajectory is (uniformly) completely observable. It is explained that LQ optimal control method is useful to design a feedback controller design of algebraically controllable and observable systems such that the actual trajectory asymptotically approaches the reference trajectory. Moreover, it is shown that the concept of algebraic controllability coincides with the concept of accessibility. Furthermore, for nonlinear mechanical control systems, a reduction condition for checking whether or not the system is algebraically controllable is provided.

## 2.1 Motivation for introducing algebraic controllability and algebraic observability

This section describes the motivation for introducing algebraic controllability and algebraic observability in this thesis. Let us consider a trajectory tracking control of the following system.

$$\dot{x} = f(x, u), \tag{2.1}$$

$$y = h(x), \tag{2.2}$$

where $x \in \boldsymbol{R}^n$, $u \in \boldsymbol{R}^m$, and $y \in \boldsymbol{R}^p$ denote state, input, and output variables, respectively. Moreover, $f : \boldsymbol{R}^n \times \boldsymbol{R}^m \to \boldsymbol{R}^n$ and $h : \boldsymbol{R}^n \to \boldsymbol{R}^p$ are meromorphic.

Here, we note that meromorphic functions are defined as elements of the quotient field of the ring of analytic functions (see appendix D).

First, we define trajectory of system (2.1)-(2.2).

**Definition 2.1** *A* **trajectory** *of system* (2.1)-(2.2) *is a pair* $(x^*(t), u^*(t))$ *satisfying*

$$\dot{x}^*(t) = f(x^*(t), u^*(t)) \quad \text{for almost all} \quad t \in \boldsymbol{R}.$$

*A trajectory* $(x^*(t), u^*(t))$ *of system* (2.1)-(2.2) *is called* **periodic** *if* $x_i^*(t)$, $u_j^*(t)$, $1 \le i \le n$, $1 \le j \le m$ *are periodic with the same period.*

Suppose that we have a trajectory $(x^*(t), u^*(t))$ of system (2.1)-(2.2). Then if we take $x(0) = x^*(0)$ and apply a feedforward control $u(t) = u^*(t)$ for all $t \ge 0$ into system (2.1)-(2.2), by the theorem on uniqueness of solution of ordinary differential equation [102], we have

$$x(t) = x^*(t) \quad \text{on} \quad \boldsymbol{R}_+.$$

Therefore, if $x^*(t)$ is a reference trajectory on $\boldsymbol{R}_+$, by taking $x(0) = x^*(0)$ and applying the feedforward control $u^*(t)$ for all $t \ge 0$, the trajectory tracking control is achieved.

However, practically it is impossible to take an initial state as $x(0) = x^*(0)$. Thus we analyze the error $x_\epsilon(t) := x(t) - x^*(t)$ between the actual trajectory $x(t)$ and the reference trajectory $x^*(t)$. Let $u_\epsilon(t) := u(t) - u^*(t)$. The error dynamics obey

$$\begin{cases} \dot{x}_\epsilon = f(x_\epsilon + x^*(t), u_\epsilon + u^*(t)) - f(x^*(t), u^*(t)), \\ y_\epsilon = h(x_\epsilon + x^*(t)) - h(x^*(t)). \end{cases} \tag{2.3}$$

If $f$ is nonlinear with respect to $x$ and $u$ variables, it is a difficult task to design a feedback control $u_\epsilon(x_\epsilon)$ such that $x_\epsilon = 0$ is stabilized without linearizing system (2.3). For the reason, we linearize system (2.3) at $(x_\epsilon, u_\epsilon) = (0, 0)$ as follows.

$$\dot{x}_\epsilon = \underbrace{\frac{\partial f}{\partial x}(x^*(t), u^*(t))}_{A(t)} x_\epsilon + \underbrace{\frac{\partial f}{\partial u}(x^*(t), u^*(t))}_{B(t)} u_\epsilon, \tag{2.4}$$

$$y_\epsilon = \underbrace{\frac{\partial h}{\partial x}(x^*(t))}_{C(t)} x_\epsilon. \tag{2.5}$$

We also say that system (2.4)-(2.5) is a linearized system of system (2.1)-(2.2) along $(x^*(t), u^*(t))$. For the linear time varying system (2.4), by using an appropriate control design technique, we can design a linear feedback control law

$u_\epsilon = K(t)x_\epsilon$ such that stabilize $x_\epsilon = 0$. In fact, it is known that if the matrices $A(t)$ and $B(t)$ are bounded on $t \in \boldsymbol{R}$, and if system (2.4) is uniformly completely controllable, then system (2.4)-(2.5) is uniformly completely stabilizable [34]. Then there exists a feedback gain $K(\cdot)$ such that the origin of closed-loop

$$\dot{x}_\epsilon = (A(t) + B(t)K(t))x_\epsilon \tag{2.6}$$

is exponentially stable [48]. Furthermore, if the origin of linear closed-loop (2.6) is exponentially stable, the origin of nonlinear closed-loop

$$\dot{x}_\epsilon = f(x_\epsilon + x^*(t), K(t)x_\epsilon + u^*(t)) - f(x^*(t), u^*(t)) \tag{2.7}$$

is locally exponentially stable [48]. Therefore if linearized system (2.4)-(2.5) is uniformly completely controllable, by applying a feedforward and an appropriate feedback control

$$u = u^*(t) + K(t)(x - x^*(t)) \tag{2.8}$$

into system (2.1)-(2.2), the actual trajectory locally exponentially approaches the reference trajectory.

To describe exactly the above, first, let us define controllability on some time interval [42].

**Definition 2.2** *System* (2.4)-(2.5) *is called* **controllable** *on* $[t_0, t_1]$ *if for all* $e \in \boldsymbol{R}^n$, *there exists a control* $u_\epsilon : \boldsymbol{R} \to \boldsymbol{R}^m$ *such that* $x_\epsilon(t_0) = e$ *and* $x_\epsilon(t_1) = 0$.

Next, let us define complete controllability [10, 42].

**Definition 2.3** *System* (2.4)-(2.5) *is called* **completely controllable** *if for all* $e \in \boldsymbol{R}^n$ *and all* $t_0 \in \boldsymbol{R}$, *there exist* $t_1 > t_0$ *and a control* $u_\epsilon : \boldsymbol{R} \to \boldsymbol{R}^m$ *such that* $x_\epsilon(t_0) = e$ *and* $x_\epsilon(t_1) = 0$.

It is known that system (2.4)-(2.5) is completely controllable if and only if for all $t_0 \in \boldsymbol{R}$, there exists $t_1 > t_0$ such that

$$W(t_0, t_1) := \int_{t_0}^{t_1} \Phi(t_0, t)B(t)B^T(t)\Phi^T(t_0, t)dt$$

is invertible [10], where $\Phi(\cdot, \cdot)$ is the transition matrix of $\dot{x}_\epsilon = A(t)x_\epsilon$. Uniform complete controllability is defined as follows [35, 42].

**Definition 2.4** *System* (2.4)-(2.5) *is called* **uniformly completely controllable** *if there exist* $\sigma > 0$ *and* $\alpha_i > 0$, $i = 1, 2, 3, 4$ *such that for all* $t \in \boldsymbol{R}$

$$\begin{cases} \alpha_1 I \le W(t, t+\sigma) \le \alpha_2 I, \\ \alpha_3 I \le \Phi(t+\sigma, t)W(t, t+\sigma)\Phi^T(t+\sigma, t) \le \alpha_4 I. \end{cases}$$

Uniform complete stabilizability is defined as follows [34].

**Definition 2.5** *System* (2.4)-(2.5) *is called* **uniformly completely stabilizable** *if for any $r > 0$, there exist a feedback gain $K(t)$ and $k > 0$ such that*

$$||\Phi_{\mathrm{cl}}(t, t_0)|| \leq k \exp(-r(t - t_0)) \quad \text{for all} \quad t_0 \in \boldsymbol{R},\ t \geq t_0,$$

*where $\Phi_{\mathrm{cl}}(\cdot, \cdot)$ be the state transition matrix of closed-loop* (2.6)*.*

Exponential stability is defined as follows [48].

**Definition 2.6** *Consider system*

$$\dot{x} = F(t, x), \tag{2.9}$$

*where $F : \boldsymbol{R} \times \boldsymbol{R}^n \to \boldsymbol{R}^n$. The origin of system* (2.9) *is called* **exponentially stable** *if there exist $c > 0$, $k > 0$, and $r > 0$ such that*

$$||x(t)|| \leq k||x(t_0)|| \exp(-r(t - t_0)) \quad \text{for all} \quad ||x(t_0)|| < c.$$

For linear system (2.6), the origin of system (2.6) is exponentially stable if and only if there exist $k > 0$ and $r > 0$ such that [48]

$$||\Phi_{\mathrm{cl}}(t, t_0)|| \leq k \exp(-r(t - t_0)) \quad \text{for all} \quad t_0 \in \boldsymbol{R},\ t \geq t_0.$$

Hence if system (2.4)-(2.5) is uniformly completely stabilizable, the origin of system (2.4)-(2.5) is exponentially stabilizable. Therefore, as the above mentioned, if system (2.4)-(2.5) is uniformly completely controllable, system (2.4)-(2.5) is exponentially stabilizable [34], and then there exists a control input such that the actual trajectory locally approaches the reference trajectory. Therefore it is important to examine whether or not linearized system (2.4)-(2.5) is uniformly completely controllable. However we can only check whether or not definition 2.4 is satisfied **only** for a **fixed** linearized system along a **specific** trajectory $(x^*(t), u^*(t))$. Hence the following questions are posed.

**Question 2.1** *What is a class of nonlinear systems* (2.1)-(2.2) *whose linearizations* (2.4)-(2.5) *along trajectories are uniformly completely controllable?*

**Question 2.2** *What is a class of trajectories stated in question 2.1?*

We will answer questions 2.1 and 2.2 in the section 2.2.

On the other hand, the available signal in system (2.1)-(2.2) might be only output signal $y$, that is, state feedback might be not available. In this case, we design a state observer called a Luenberger type observer

$$\dot{\hat{x}}_\epsilon = A(t)\hat{x}_\epsilon + B(t)u_\epsilon + L(t)(y_\epsilon - C(t)\hat{x}_\epsilon). \tag{2.10}$$

Let $e := x_\epsilon - \hat{x}_\epsilon$. By Eqs. (2.4)-(2.5) and (2.10), the dynamics of $e$ obey

$$\dot{e} = (A(t) - L(t)C(t))\, e. \tag{2.11}$$

It is known [35] that if system (2.4)-(2.5) is uniformly completely observable, then there exists an observer gain $L(\cdot)$ such that the origin of system (2.11) is exponentially stable. Moreover, it is known [35] that if system (2.4)-(2.5) is uniformly completely controllable and uniformly completely observable, then there exist a feedback gain $K(\cdot)$ and an observer gain $L(\cdot)$ such that the origin of closed-loop

$$\begin{pmatrix} \dot{x}_\epsilon \\ \dot{\hat{x}}_\epsilon \end{pmatrix} = \begin{pmatrix} A(t) & B(t)K(t) \\ L(t)C(t) & A(t) - L(t)C(t) + B(t)K(t) \end{pmatrix} \begin{pmatrix} x_\epsilon \\ \hat{x}_\epsilon \end{pmatrix}$$

is exponentially stable. Therefore if system (2.4)-(2.5) is uniformly completely controllable and uniformly completely observable, then by applying a feedforward and an appropriate error-state estimate feedback control

$$u = u^*(t) + K(t)\hat{x}_\epsilon$$

into system (2.1)-(2.2), it is expected that the actual trajectory locally exponentially approaches the reference trajectory $x^*(t)$.

In order to design an appropriate observer gain $L(\cdot)$, we also need the concept of uniform complete observability [42]. First, let us define observability on $[t_0, t_1]$.

**Definition 2.7** *System* (2.4)-(2.5) *is called* **observable** *on* $[t_0, t_1]$ *if for all present state* $x_\epsilon(t_1) \in \boldsymbol{R}^n$ *can be uniquely determined by* $(y_\epsilon(t), u_\epsilon(t))$ *on* $[t_0, t_1]$.

Next, let us define complete observability [10].

**Definition 2.8** *System* (2.4)-(2.5) *is called* **completely observable** *if for all* $t_1 \in \boldsymbol{R}$, *there exists* $t_0 < t_1$ *such that all present state* $x_\epsilon(t_1) \in \boldsymbol{R}^n$ *can be uniquely determined by* $(y_\epsilon(t), u_\epsilon(t))$ *on* $[t_0, t_1]$.

It is known that system (2.4)-(2.5) is completely observable if and only if for all $t_1 \in \boldsymbol{R}$, there exists $t_0 < t_1$ such that

$$M(t_1, t_0) := \int_{t_0}^{t_1} \Phi^T(t, t_1) C^T(t) C(t) \Phi(t, t_1) dt$$

is invertible [10]. Uniform complete observability is defined as follows [42].

**Definition 2.9** *System* (2.4)-(2.5) *is called* **uniformly completely observable** *if there exist* $\sigma > 0$ *and* $\alpha_i > 0$, $i = 1, 2, 3, 4$ *such that for all* $t \in \boldsymbol{R}$,

$$\begin{cases} \alpha_1 I \leq M(t, t-\sigma) \leq \alpha_2 I, \\ \alpha_3 I \leq \Phi^T(t, t-\sigma) M(t, t-\sigma) \Phi(t, t-\sigma) \leq \alpha_4 I. \end{cases}$$

As the above mentioned, it is important to verify whether or not linearized system (2.4)-(2.5) is uniformly completely observable and the following questions are posed for the same reason as questions 2.1 and 2.2.

**Question 2.3** *What is a class of nonlinear systems* (2.1)-(2.2) *whose linearizations* (2.4)-(2.5) *along trajectories are uniformly completely observable?*

**Question 2.4** *What is a class of trajectories stated in question 2.3?*

We will answer questions 2.3 and 2.4 in the section 2.3.

**Remark 2.1** *For a special class of nonlinear systems called nonholonomic chained systems, reference [89] has given a sufficient condition for certain linearizations of nonlinear systems in the class to be uniformly completely controllable and uniformly completely observable.* ■

In the above discussion, it is significant that a given reference trajectory composes of a trajectory of system (2.1)-(2.2). Unfortunately, in general, it is difficult to verify whether or not a given reference trajectory composes of a trajectory of system (2.1)-(2.2) because the vector field $f$ is nonlinear with respect to $x$ and $u$. However, fortunately, if a given nonlinear system (2.1)-(2.2) is differentially flat, we can easily verify that. Differential flatness of system (2.1)-(2.2) is defined as follows [22] (see appendix E).

**Definition 2.10** *System* (2.1)-(2.2) *is called* **differentially flat** *if there exist smooth mappings* $\phi_1 : \boldsymbol{R}^m \times \boldsymbol{R}^m \times \cdots \to \boldsymbol{R}^n$, $\phi_2 : \boldsymbol{R}^m \times \boldsymbol{R}^m \times \cdots \to \boldsymbol{R}^m$, *and* $\psi : \boldsymbol{R}^n \times (\boldsymbol{R}^m \times \cdots) \to \boldsymbol{R}^m$ *depending only on a finite number of variables, respectively, such that*

$$v := \psi(x, u, \dot{u}, \cdots) \Rightarrow \begin{pmatrix} x \\ u \end{pmatrix} = \begin{pmatrix} \phi_1(v, \dot{v}, \ddot{v}, \cdots) \\ \phi_2(v, \dot{v}, \ddot{v}, \cdots) \end{pmatrix}.$$

*In addition, if system* (2.1)-(2.2) *is differentially flat, the variable* $v$ *satisfying the above condition is called a* **flat output** *of system* (2.1)-(2.2).

Assume that system (2.1)-(2.2) is differentially flat with a flat output $v$. Then there exist smooth mappings $\phi_1 : \boldsymbol{R}^m \times \boldsymbol{R}^m \times \cdots \to \boldsymbol{R}^n$ and $\phi_2 : \boldsymbol{R}^m \times \boldsymbol{R}^m \times \cdots \to \boldsymbol{R}^m$ such that

$$\begin{pmatrix} x \\ u \end{pmatrix} = \begin{pmatrix} \phi_1(v, \dot{v}, \cdots) \\ \phi_2(v, \dot{v}, \cdots) \end{pmatrix}.$$

Now assume that $v(t)$ has been defined on $\boldsymbol{R}$. Taking an initial state $x(0) = \phi_1(v(0), \dot{v}(0), \cdots)$ and applying feedforward control $u(t) = \phi_2(v(t), \dot{v}(t), \cdots)$ on

$\boldsymbol{R}$ to system (2.1)-(2.2), by the theorem on uniqueness of solution of ordinary differential equation [102], we have

$$x(t) = \phi_1(v(t), \dot{v}(t), \cdots) \quad \text{on} \quad \boldsymbol{R}.$$

Therefore if we consider a reference trajectory of system (2.1)-(2.2) as $\phi_1(v(t), \dot{v}(t), \cdots)$, the trajectory $(x(t), u(t)) = (\phi_1(v(t), \dot{v}(t), \cdots), \phi_2(v(t), \dot{v}(t), \cdots))$ is . We will show how to apply differentially flat property in example 2.38 in the next section.

## 2.2 Algebraic controllability

In order to answer the questions 2.1 and 2.2, this section introduces novel concepts called algebraic controllability and controllable trajectory of system (2.1)-(2.2). It is shown that if a given nonlinear system (2.1)-(2.2) is algebraically controllable, then every linearized system along any (periodic) controllable trajectory is (uniformly) completely controllable.

First, we give some preliminaries in order to define algebraic controllability. Let $\mathcal{M}_{(x,u)}$ denote the field of all meromorphic functions depending on a finite number of variables of

$$\left\{ x_i, u_k^{(l)} \mid 1 \leq i \leq n, 1 \leq k \leq m, l \geq 0 \right\}.$$

The field $\mathcal{M}_{(x,u)}$ can be endowed with a differential structure determined by Eq. (2.1) as follows:

$$\dot{\phi}(x, u, \dot{u}, \cdots) := \frac{\partial \phi}{\partial x} f(x, u) + \sum_{l \geq 0} \frac{\partial \phi}{\partial u^{(l)}} u^{(l+1)},$$

where $\phi(x, u, \dot{u}, \cdots) \in \mathcal{M}_{(x,u)}$. Thus $M_{(x,u)}$ is a differential field (see appendix C). A vector space $\mathcal{E}_{(x,u)}$ of differential one-forms spanned over $\mathcal{M}_{(x,u)}$ is defined [16] as

$$\mathcal{E}_{(x,u)} := \operatorname{span}_{\mathcal{M}_{(x,u)}} \left\{ dx_i, du_k^{(l)} \mid 1 \leq i \leq n, 1 \leq k \leq m, l \geq 0 \right\}.$$

Then for any $\phi \in \mathcal{M}_{(x,u)}$, differential $d : \mathcal{M}_{(x,u)} \to \mathcal{E}_{(x,u)}$ is defined as

$$d\phi := \frac{\partial \phi}{\partial x} dx + \sum_{l \geq 0} \frac{\partial \phi}{\partial u^{(l)}} du^{(l)}. \tag{2.12}$$

Let $\mathcal{D}_{(x,u)} := \mathcal{M}_{(x,u)} \left[ \frac{d}{dt} \right]$. For $\alpha = \sum_{i=0}^{m} \alpha_i \frac{d^i}{dt^i} \in \mathcal{D}_{(x,u)}$, $\alpha_i \in \mathcal{M}_{(x,u)}$, $\frac{d}{dt}\alpha$ is defined as

$$\frac{d}{dt}\alpha := \sum_{i=0}^{m} \left( \alpha_i \frac{d^{i+1}}{dt^{i+1}} + \dot{\alpha}_i \frac{d^i}{dt^i} \right).$$

Hence $\mathcal{D}_{(x,u)}$ is a left skew polynomial ring, and thus elements of $\mathcal{D}_{(x,u)}$ can act on the vector space $\mathcal{E}_{(x,u)}$ (see appendix C), that is, the vector space $\mathcal{E}_{(x,u)}$ can be endowed with a differential structure by defining a derivative operator $\frac{d}{dt}$ as follows:

$$\frac{d}{dt}\left(\sum_{i=1}^{n} a_i dx_i + \sum_{l\geq 0}\sum_{k=1}^{m} c_{k,l} du_k^{(l)}\right) := \sum_{i=1}^{n}\left(\dot{a}_i dx_i + a_i \sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j} dx_i\right) + \sum_{l\geq 0}\sum_{k=1}^{m}(\dot{c}_{k,l} du_k^{(l)} + c_{k,l} du_k^{(l+1)}),$$

where $\sum_{i=1}^{n} a_i dx_i + \sum_{l\geq 0}\sum_{k=1}^{m} c_{k,l} du_k^{(l)} \in \mathcal{E}_{(x,u)}$. More generally, see appendix C. Furthermore, $\mathcal{D}_{(x,u)}$ is simple and a non-commutative Euclidean domain (see proposition B.3 in appendix B). Thus since $\mathcal{D}_{(x,u)}$ is a left and right principal ideal domain [15], $\mathcal{D}_{(x,u)}$ has the left and right Ore property (see proposition A.2 in appendix A). Thus, $\mathcal{D}_{(x,u)}$ admits a skew field $\mathcal{K}_{(x,u)}$ of fractions containing elements of the form $k = r^{-1}n$ or $k = nr^{-1}$, where $0 \neq r \in \mathcal{D}_{(x,u)}$ and $n \in \mathcal{D}_{(x,u)}$ (see proposition A.3 in appendix A). Hence, the rank of a matrix $R_{(x,u)} \in \mathcal{D}_{(x,u)}^{a\times b}$ is well defined as $\operatorname{rank} R_{(x,u)} := \dim\left(\mathcal{K}_{(x,u)}^{1\times a} R_{(x,u)}\right) = \dim\left(R_{(x,u)}\mathcal{K}_{(x,u)}^{b}\right)$.

**Definition 2.11** *A matrix* $U \in \mathcal{D}_{(x,u)}^{a\times a}$ *is called* **unimodular** *if there exists a matrix* $U^{-1} \in \mathcal{D}_{(x,u)}^{a\times a}$ *with* $UU^{-1} = U^{-1}U = I_a$.

The following proposition [15] is important to give the definition of algebraic controllability (see proposition B.4 and lemma B.5 in appendix B.3).

**Proposition 2.1** *Suppose that* $R_{(x,u)} \in \mathcal{D}_{(x,u)}^{a\times b}$. *Then there exist unimodular matrices* $U_{(x,u)} \in \mathcal{D}_{(x,u)}^{a\times a}$ *and* $V_{(x,u)} \in \mathcal{D}_{(x,u)}^{b\times b}$ *such that*

$$U_{(x,u)}R_{(x,u)}V_{(x,u)} = \begin{pmatrix} \operatorname{diag}(1,\cdots,1,\alpha) & 0 \\ 0 & 0 \end{pmatrix}, \tag{2.13}$$

*where* $0 \neq \alpha \in \mathcal{D}_{(x,u)}$, *and where* $s := \operatorname{rank} R_{(x,u)}$. *Moreover, the degree of the polynomial* $\alpha$ *is constant for any unimodular matrices* $U_{(x,u)}$ *and* $V_{(x,u)}$ *satisfying* (2.13).

**Remark 2.2** *The above normal form is called the Jacobson form [15]. Since* $\mathcal{D}_{(x,u)}$ *is Euclidean [15], the matrices* $U_{(x,u)}$ *and* $V_{(x,u)}$ *can be obtained by repeating* **elementary row and column operations** *for the matrix* $R_{(x,u)}$. *Here, elementary row (column) operations are defined as follows:*

1. *Interchange row (column)* $i$ *and row (column)* $j$.

2. *To row (column)* $i$ *add* $d \in \mathcal{D}_{(x,u)}$ *times row (column)* $j$, $i \neq j$.

*3. Multiply row (column) $i$ by a non-zero element in $\mathcal{M}_{(x,u)}$.*

*Each elementary row (column) operation on a matrix corresponds to the left (right) multiplication of the matrix by an appropriate unimodular matrix.*

*Moreover in order to obtain the Jacobson form of the matrix $R_{(x,u)}$, for example, we can use symbolic packages of computer algebra such as SINGULAR [25] and OreModules [14].* ■

To define algebraic controllability and algebraic observability, we define hyper-regularity [65, 66].

**Definition 2.12** *Let $R_{(x,u)} \in \mathcal{D}^{a\times b}_{(x,u)}$. The matrix $R_{(x,u)}$ is called* **hyper-regular** *if there exist unimodular matrices $U_{(x,u)} \in \mathcal{D}^{a\times a}_{(x,u)}$ and $V_{(x,u)} \in \mathcal{D}^{b\times b}_{(x,u)}$ such that*

$$\begin{aligned} U_{(x,u)}R_{(x,u)}V_{(x,u)} &= \begin{pmatrix} I_a & 0 \end{pmatrix} \quad \text{if} \quad a \leq b, \\ U_{(x,u)}R_{(x,u)}V_{(x,u)} &= \begin{pmatrix} I_b \\ 0 \end{pmatrix} \quad \text{if} \quad a \geq b. \end{aligned}$$

From now on, we define algebraic controllability and controllable trajectory. First, differentiating both sides of Eq. (2.1), we have

$$P^c_{(x,u)} \begin{pmatrix} dx \\ du \end{pmatrix} = 0, \tag{2.14}$$

where

$$P^c_{(x,u)} := \begin{pmatrix} \frac{d}{dt}I - \frac{\partial f}{\partial x}(x,u) & -\frac{\partial f}{\partial u}(x,u) \end{pmatrix}. \tag{2.15}$$

Since $f$ is meromorphic with respect to each variable, coefficients of polynomials of each element of $P^c_{(x,u)}$ are meromorphic functions. Thus $P^c_{(x,u)} \in \mathcal{D}^{n\times(n+m)}_{(x,u)}$. If we can transform the matrix $P^c_{(x,u)}$ defined by (2.15) into the simplest form of the Jacobson form, we say that system (2.1)-(2.2) is algebraically controllable.

**Definition 2.13** *System* (2.1)-(2.2) *is called* **algebraically controllable** *if $P^c_{(x,u)}$ defined by* (2.15) *is hyper-regular, that is, there exist unimodular matrices $U_{(x,u)} \in \mathcal{D}^{n\times n}_{(x,u)}$ and $V_{(x,u)} \in \mathcal{D}^{(n+m)\times(n+m)}_{(x,u)}$ satisfying*

$$U_{(x,u)}P^c_{(x,u)}V_{(x,u)} = \begin{pmatrix} I_n & 0 \end{pmatrix}. \tag{2.16}$$

**Remark 2.3** *Algebraic controllability is a necessary condition for differential flatness ( see proposition 3 in [66]). Furthermore, system* (2.1)-(2.2) *is algebraically controllable if and only if system* (2.1)-(2.2) *is accessible [16] (see section 2.6).* ■

**Remark 2.4** *As a special system of system* (2.1), *let us consider*

$$\dot{x}^1 = f^1(x^1, u^1), \tag{2.17}$$
$$\dot{x}^2 = f^2(x^2, u^2), \tag{2.18}$$

*where* $x_1 \in \boldsymbol{R}^{n_1}$, $x_2 \in \boldsymbol{R}^{n_2}$ *are state variables and* $u_1 \in \boldsymbol{R}^{m_1}$, $u_2 \in \boldsymbol{R}^{m_2}$ *are input variables. Furthermore,* $f^1$ *and* $f^2$ *are meromorphic with respect to each variable. Then if subsystem* (2.17) *and subsystem* (2.18) *are both algebraically controllable, system* (2.17)-(2.18) *is algebraically controllable. In fact, differentiating both sides of* (2.17)-(2.18), *we have*

$$\underbrace{\begin{pmatrix} P^1 & 0 \\ 0 & P^2 \end{pmatrix}}_{P} \begin{pmatrix} dx^1 \\ du^1 \\ dx^2 \\ du^2 \end{pmatrix} = 0,$$

*where*

$$P^1 := \begin{pmatrix} \frac{d}{dt} I_{n_1} - \frac{\partial f^1}{\partial x^1} & -\frac{\partial f^1}{\partial u^1} \end{pmatrix},$$
$$P^2 := \begin{pmatrix} \frac{d}{dt} I_{n_2} - \frac{\partial f^2}{\partial x^2} & -\frac{\partial f^2}{\partial u^2} \end{pmatrix}.$$

*Since subsystem* (2.17) *is algebraically controllable, there exist unimodular matrices* $U^1$ *and* $V^1$ *such that* $U^1 P^1 V^1 = \begin{pmatrix} I_{n_1} & 0 \end{pmatrix}$. *Hence*

$$\begin{pmatrix} U^1 & 0 \\ 0 & I_{n_2} \end{pmatrix} P \begin{pmatrix} V^1 & 0 \\ 0 & I_{n_2+m_2} \end{pmatrix} = \begin{pmatrix} I_{n_1} & 0 & 0 \\ 0 & 0 & P_{(2)} \end{pmatrix} \tag{2.19}$$

*On the other hand, since system* (2.18) *is algebraically controllable, there exist unimodular matrices* $U^2$ *and* $V^2$ *such that* $U^2 P^2 V^2 = \begin{pmatrix} I_{n_2} & 0 \end{pmatrix}$. *Therefore by* (2.19), *there exist unimodular matrices* $U$ *and* $V$ *such that* $UPV = \begin{pmatrix} I_{n_1+n_2} & 0 \end{pmatrix}$. *Hence system* (2.17)-(2.18) *is algebraically controllable.*

*In section 2.8, we study another system which has a more special structure.* ■

Note that if system (2.1)-(2.2) is linear, algebraic controllability is equivalent to controllability in the usual sense. In fact, let us consider linear time invariant systems

$$\dot{x} = Ax + Bu, \tag{2.20}$$
$$y = Cx, \tag{2.21}$$

where $A \in \boldsymbol{R}^{n \times n}$, $B \in \boldsymbol{R}^{n \times m}$, and $C \in \boldsymbol{R}^{p \times n}$ are constant matrices. Then differentiating both sides of (2.20), we have

$$\underbrace{\begin{pmatrix} \frac{d}{dt} I_n - A & -B \end{pmatrix}}_{P^c_{\text{linear}}} \begin{pmatrix} dx \\ du \end{pmatrix} = 0.$$

We note that $P^c_{\text{linear}} \in \left(\boldsymbol{R}[\frac{d}{dt}]\right)^{n\times(n+m)} \subset \mathcal{D}^{n\times(n+m)}_{(x,u)}$. Thus we can transform the matrix $P^c_{\text{linear}}$ into the Jacobson form. Actually, there exist unimodular matrices $U \in \left(\boldsymbol{R}[\frac{d}{dt}]\right)^{n\times n}$ and $V \in \left(\boldsymbol{R}[\frac{d}{dt}]\right)^{(n+m)\times(n+m)}$ such that

$$UP^c_{\text{linear}}V = \begin{pmatrix}\Delta & 0\end{pmatrix},$$

where $\Delta := \text{diag}(1,\cdots,1,\alpha)$ and $0 \neq \alpha \in \boldsymbol{R}[\frac{d}{dt}]$. Since $\boldsymbol{R}[\frac{d}{dt}]$ is a commutative ring, this form is called the **Smith form** [88]. On this account, in the case of linear time invariant systems, we should say that system (2.20)-(2.21) is algebraically controllable if there exists unimodular matrices $U \in \left(\boldsymbol{R}[\frac{d}{dt}]\right)^{n\times n}$ and $V \in \left(\boldsymbol{R}[\frac{d}{dt}]\right)^{(n+m)\times(n+m)}$ such that

$$UP^c_{\text{linear}}V = \begin{pmatrix}I_n & 0\end{pmatrix}. \tag{2.22}$$

It is known [88] that system (2.20)-(2.21) is controllable if and only if there exist unimodular matrices $U \in \left(\boldsymbol{R}[\frac{d}{dt}]\right)^{n\times n}$ and $V \in \left(\boldsymbol{R}[\frac{d}{dt}]\right)^{(n+m)\times(n+m)}$ satisfying (2.22). Therefore system (2.20)-(2.21) is algebraically controllable if and only if system (2.20)-(2.21) is controllable.

Algebraic controllability is invariant under an analytic coordinate transformation.

**Theorem 2.2** *Suppose that system* (2.1)-(2.2) *is algebraically controllable. Moreover, suppose that $x = \phi(\hat{x})$ is an analytic coordinate transformation with the analytic inverse. Then the transformed system*

$$\dot{\hat{x}} = \left(\frac{\partial\phi}{\partial\hat{x}}\right)^{-1} f(\phi(\hat{x}),u), \tag{2.23}$$

$$y = h(\phi(\hat{x})) \tag{2.24}$$

*is also algebraically controllable.*

**Proof** Differentiating both sides of (2.23), we have

$$\underbrace{\begin{pmatrix}\frac{d}{dt}I - \frac{\partial}{\partial\hat{x}}\left[A(\hat{x})^{-1}f(\phi(\hat{x}),u)\right] & -A(\hat{x})^{-1}\frac{\partial f}{\partial u}\end{pmatrix}}_{\hat{P}^c_{(\hat{x},u)}}\begin{pmatrix}d\hat{x}\\ du\end{pmatrix} = 0,$$

where $A(\hat{x}) := \frac{\partial\phi}{\partial\hat{x}}(\hat{x})$. Since

$$\frac{\partial}{\partial\hat{x}_i}\left[A(\hat{x})^{-1}f(\phi(\hat{x}),u)\right] = -A(\hat{x})^{-1}\frac{\partial A}{\partial\hat{x}_i}\dot{\hat{x}} + A(\hat{x})^{-1}\frac{\partial f}{\partial x}\frac{\partial\phi}{\partial\hat{x}_i},$$

by a direct calculation, we obtain

$$\hat{P}^c_{(\hat{x},u)} = A(\hat{x})^{-1}\left(A(\hat{x})\frac{d}{dt}I + \begin{pmatrix}\frac{\partial A}{\partial\hat{x}_1}\dot{\hat{x}} & \cdots & \frac{\partial A}{\partial\hat{x}_n}\dot{\hat{x}}\end{pmatrix} - \frac{\partial f}{\partial x}A(\hat{x}) \quad -\frac{\partial f}{\partial u}\right).$$

Furthermore, since

$$\frac{\partial a_{ij}}{\partial \hat{x}_k} = \frac{\partial^2 \phi_i}{\partial \hat{x}_j \partial \hat{x}_k} = \frac{\partial a_{ik}}{\partial \hat{x}_j},$$

we have

$$\begin{aligned}
\begin{pmatrix} \frac{\partial A}{\partial \hat{x}_1}\dot{\hat{x}} & \cdots & \frac{\partial A}{\partial \hat{x}_n}\dot{\hat{x}} \end{pmatrix} &= \begin{pmatrix} \frac{\partial a_{11}}{\partial \hat{x}_1}\dot{\hat{x}}_1 + \cdots + \frac{\partial a_{1n}}{\partial \hat{x}_1}\dot{\hat{x}}_n & \cdots & \frac{\partial a_{11}}{\partial \hat{x}_1}\dot{\hat{x}}_n + \cdots + \frac{\partial a_{1n}}{\partial \hat{x}_n}\dot{\hat{x}}_n \\ \vdots & & \vdots \\ \frac{\partial a_{n1}}{\partial \hat{x}_1}\dot{\hat{x}}_1 + \cdots + \frac{\partial a_{nn}}{\partial \hat{x}_1}\dot{\hat{x}}_n & \cdots & \frac{\partial a_{n1}}{\partial \hat{x}_n}\dot{\hat{x}}_n + \cdots + \frac{\partial a_{nn}}{\partial \hat{x}_n}\dot{\hat{x}}_n \end{pmatrix} \\
&= \begin{pmatrix} \frac{\partial a_{11}}{\partial \hat{x}_1}\dot{\hat{x}}_1 + \cdots + \frac{\partial a_{11}}{\partial \hat{x}_n}\dot{\hat{x}}_n & \cdots & \frac{\partial a_{1n}}{\partial \hat{x}_1}\dot{\hat{x}}_n + \cdots + \frac{\partial a_{1n}}{\partial \hat{x}_n}\dot{\hat{x}}_n \\ \vdots & & \vdots \\ \frac{\partial a_{n1}}{\partial \hat{x}_1}\dot{\hat{x}}_1 + \cdots + \frac{\partial a_{n1}}{\partial \hat{x}_n}\dot{\hat{x}}_n & \cdots & \frac{\partial a_{nn}}{\partial \hat{x}_1}\dot{\hat{x}}_n + \cdots + \frac{\partial a_{nn}}{\partial \hat{x}_n}\dot{\hat{x}}_n \end{pmatrix} \\
&= \dot{A}(\hat{x}).
\end{aligned}$$

Therefore

$$\hat{P}^c_{(\hat{x},u)} = A(\hat{x})^{-1} P^c_{(\phi(x),u)} \begin{pmatrix} A(\hat{x}) & 0 \\ 0 & I \end{pmatrix}.$$

On the other hand, since system (2.1)-(2.2) is algebraically controllable, there exist unimodular matrices $U_{(x,u)} \in \mathcal{D}^{n\times n}_{(x,u)}$ and $V_{(x,u)} \in \mathcal{D}^{(n+m)\times(n+m)}_{(x,u)}$ satisfying (2.16). Hence we have

$$\underbrace{\left(U_{(\phi(\hat{x}),u)} A(\hat{x})\right)}_{\hat{U}_{(\hat{x},u)}} \hat{P}^c_{(\hat{x},u)} \underbrace{\left(\begin{pmatrix} A(\hat{x})^{-1} & 0 \\ 0 & I \end{pmatrix} V_{(\phi(\hat{x}),u)}\right)}_{\hat{V}_{(\hat{x},u)}} = \begin{pmatrix} I_n & 0 \end{pmatrix}.$$

Since $\hat{U}_{(\hat{x},u)}$ and $\hat{V}_{(\hat{x},u)}$ are unimodular, system (2.23)-(2.24) is algebraically controllable. □

As a corollary of theorem 2.2, algebraic controllability is invariant under a linear coordinate transformation.

**Corollary 2.3** *Suppose that system* (2.1)-(2.2) *is algebraically controllable. Let* $x = A\tilde{x}$, *where* $A \in \boldsymbol{R}^{n\times n}$ *is invertible. Then the transformed system*

$$\begin{aligned} \dot{\tilde{x}} &= A^{-1} f(A\tilde{x}, u), \\ y &= h(A\tilde{x}) \end{aligned}$$

*is also algebraically controllable.*

Moreover, algebraic controllability is invariant under a static feedback.

**Theorem 2.4** *Suppose that system* (2.1)-(2.2) *is algebraically controllable. Moreover, suppose that* $u = \psi(x, v)$ *is an analytic static feedback and* $\det \frac{\partial \psi}{\partial v}(x, v) \not\equiv 0$, *where* $v(t) \in \boldsymbol{R}^m$ *is a new input variable. Then the resulting system*

$$\dot{x} = f(x, \psi(x, v)) \tag{2.25}$$
$$y = h(x) \tag{2.26}$$

*is also algebraically controllable.*

**Proof** Differentiating both sides of (2.25)-(2.26), we have

$$\underbrace{\begin{pmatrix} \frac{d}{dt}I - \frac{\partial f}{\partial x}(x, \psi(x, v)) - \frac{\partial f}{\partial u}(x, \psi(x, v))\frac{\partial \psi}{\partial x}(x, v) & -\frac{\partial f}{\partial u}(x, \psi(x, v))\frac{\partial \psi}{\partial v}(x, v) \end{pmatrix}}_{\hat{P}^c_{(x,v)}} \begin{pmatrix} dx \\ dv \end{pmatrix} = 0.$$

By a straightforward calculation, we get

$$\hat{P}^c_{(x,v)} = P^c_{(x,\psi(x,v))} \begin{pmatrix} I & 0 \\ \frac{\partial \psi}{\partial x} & I \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & \frac{\partial \psi}{\partial v} \end{pmatrix}.$$

On the other hand, since system (2.1)-(2.2) is algebraically controllable, there exist unimodular matrices $U_{(x,u)} \in \mathcal{D}^{n\times n}_{(x,u)}$ and $V_{(x,u)} \in \mathcal{D}^{(n+m)\times(n+m)}_{(x,u)}$ satisfying (2.16). Therefore

$$\underbrace{U_{(x,\psi(x,v))}}_{\hat{U}_{(x,v)}} \hat{P}^c_{(x,v)} \underbrace{\left[ \begin{pmatrix} I & 0 \\ 0 & (\frac{\partial \psi}{\partial v})^{-1} \end{pmatrix} \begin{pmatrix} I & 0 \\ -\frac{\partial \psi}{\partial x} & I \end{pmatrix} V_{(x,\psi(x,v))} \right]}_{\hat{V}_{(x,v)}} = \begin{pmatrix} I_n & 0 \end{pmatrix}.$$

Since $\hat{U}_{(x,v)}$ and $\hat{V}_{(x,v)}$ are unimodular, system (2.25)-(2.26) is also algebraically controllable. □

By theorems 2.2 and 2.4, we have the following corollary.

**Corollary 2.5** *Suppose that system* (2.1)-(2.2) *is algebraically controllable. Moreover, suppose that* $x = \phi(\hat{x})$ *is an analytic coordinate transformation with the analytic inverse,* $u = \psi(\hat{x}, v)$ *is an analytic static feedback, and* $\det \frac{\partial \psi}{\partial v}(x, v) \not\equiv 0$, *where* $v(t) \in \boldsymbol{R}^m$ *is a new input variable. Then the resulting system*

$$\dot{\hat{x}} = \left(\frac{\partial \phi}{\partial \hat{x}}\right)^{-1} f(\phi(\hat{x}), \psi(\hat{x}, v)),$$
$$y = h(\phi(\hat{x}))$$

*is also algebraically controllable.*

In order to relate algebraic controllability and uniform complete controllability, we define controllable trajectory of system (2.1)-(2.2). To define controllable trajectory, we need the following definition.

**Definition 2.14** *Let $R_{(x,u)} \in \mathcal{D}^{a\times b}_{(x,u)}$ and let $(x^*(t), u^*(t)) \in \boldsymbol{R}^n \times \boldsymbol{R}^m$ be a trajectory of system (2.1)-(2.2). The matrix $R_{(x^*(t),u^*(t))}$ is defined by substituting $x^*(t)$ and $u^*(t)$ into $x$ and $u$ in each polynomial element of $R_{(x,u)}$, respectively. Furthermore, the matrix $R_{(x^*(t),u^*(t))}$ is called* **bounded** *if every coefficient function of each polynomial element of $R_{(x^*(t),u^*(t))}$ is bounded on $\boldsymbol{R}$.*

When we use a variational method to generate a trajectory, we often get a piecewise smooth trajectory. For example, see chapter 4. Hence, we herein define a set of piecewise smooth functions. Let $C^\infty_{\text{a.e.}}$ be the set of all functions which are smooth except for a countable set of exception points $\boldsymbol{E}(a) \subset \boldsymbol{R}$ for each $a \in C^\infty_{\text{a.e.}}$, that is, for each $a \in C^\infty_{\text{a.e.}}$ there exists a countable set $\boldsymbol{E}(a) \subset \boldsymbol{R}$ such that $a \in C^\infty(\boldsymbol{R}\backslash\boldsymbol{E}(a), \boldsymbol{R})$. For example, the following functions are constrained in $C^\infty_{\text{a.e.}}$.

- $\frac{1}{t}$.
- $\begin{cases} t+1 & (t \le 0), \\ t-1 & (t > 0). \end{cases}$
- $\begin{cases} -t & (t < -1), \\ t+1 & (-1 \le t < 0), \\ e^{-t} & (t \ge 0). \end{cases}$

For each $a \in C^\infty_{\text{a.e.}}$, a time derivative $\dot{a}$ is defined on $\boldsymbol{R}\backslash\boldsymbol{E}(a)$ in the usual sense. We do not define the value $\dot{a}(t)$ for $t \in \boldsymbol{E}(a)$. For example,

$$a(t) := \begin{cases} -t & (t < -1), \\ t+1 & (-1 \le t < 0), \\ e^{-t} & (t \ge 0). \end{cases} \quad \Rightarrow \dot{a}(t) = \begin{cases} -1 & (t < -1), \\ 1 & (-1 < t < 0), \\ -e^{-t} & (t > 0). \end{cases}$$

For $a, b \in C^\infty_{\text{a.e.}}$, $a+b$ and $a \cdot b$ are defined on $\boldsymbol{R}\backslash(\boldsymbol{E}(a) \cup \boldsymbol{E}(b))$ as follows.

$$(a+b)(t) := a(t) + b(t),$$
$$(a \cdot b)(t) := a(t) \cdot b(t).$$

For $\alpha = \sum_{i=0}^{k} \alpha_i \frac{d^i}{dt^i} \in C^\infty_{\text{a.e.}}[\frac{d}{dt}]$, $\alpha_i \in C^\infty_{\text{a.e.}}$, $\frac{d}{dt}\alpha$ is defined as

$$\frac{d}{dt}\alpha := \sum_{i=0}^{k} \left( \alpha_i \frac{d^{i+1}}{dt^{i+1}} + \dot{\alpha}_i \frac{d^i}{dt^i} \right).$$

We define a set of **piecewise smooth functions** defined for all $t \in \boldsymbol{R}$ as

$$C^\infty_{\mathrm{pw}} := \{a \in C^\infty_{\mathrm{a.e.}} \mid a(t) \text{ is defined for all } t \in \boldsymbol{R}\}.$$

For example, although $\frac{1}{t} \notin C^\infty_{\mathrm{pw}}$, $\begin{cases} \frac{1}{t} & \text{for}\, t \in \boldsymbol{R}\backslash\{0\} \\ 0 & \text{for}\, t = 0 \end{cases}$ is contained in $C^\infty_{\mathrm{pw}}$. For an algebraically controllable system (2.1)-(2.2), controllable trajectory is composed of functions in $C^\infty_{\mathrm{pw}}$.

**Definition 2.15** *Suppose that system* (2.1)-(2.2) *is algebraically controllable. A trajectory* $(x^*(t), u^*(t))$ *of system* (2.1)-(2.2) *is called a* **controllable trajectory** *if the following conditions are satisfied:*

1. $(x^*, u^*) \in (C^\infty_{\mathrm{pw}})^n \times (C^\infty_{\mathrm{pw}})^m$.
2. *The matrix* $P^c_{(x^*(t),u^*(t))}$ *is bounded.*
3. *There exist unimodular matrices* $U_{(x,u)} \in \mathcal{D}^{n\times n}_{(x,u)}$ *and* $V_{(x,u)} \in \mathcal{D}^{(n+m)\times(n+m)}_{(x,u)}$ *satisfying* (2.16) *such that*

$$U_{(x^*(t),u^*(t))}, U^{-1}_{(x^*(t),u^*(t))} \in \left(C^\infty_{\mathrm{a.e.}}\left[\frac{d}{dt}\right]\right)^{n\times n},$$
$$V_{(x^*(t),u^*(t))}, V^{-1}_{(x^*(t),u^*(t))} \in \left(C^\infty_{\mathrm{a.e.}}\left[\frac{d}{dt}\right]\right)^{(n+m)\times(n+m)}.$$

**Remark 2.5** *"The matrix* $P^c_{(x^*(t),u^*(t))}$ *is bounded" means that each element of* $A(t)$ *and* $B(t)$ *defined in* (2.4) *is bounded on* $\boldsymbol{R}$. ■

**Remark 2.6** *If a given system* (2.1)-(2.2) *is algebraically controllable, there exist unimodular matrices* $U_{(x,u)}$ *and* $V_{(x,u)}$ *satisfying* (2.16). *However, the matrices are not unique. For the reason, we have defined controllable trajectory such as definition 2.15.* ■

**Remark 2.7** *If a given system* (2.1)-(2.2) *is differentially flat, we can easily find a controllable trajectory of system* (2.1)-(2.2). ■

In the case of linear time invariant systems (2.20)-(2.21), the concept of controllability coincides with the concept of complete controllability [42]. Hence if we consider linear time invariant systems (2.20)-(2.21), in order to characterize algebraic controllability, the concept of controllable trajectory is not required. Furthermore, in [104], it has been shown that system (2.20)-(2.21) is linearly flat if and only if there exist unimodular matrices $U \in \left(\boldsymbol{R}[\frac{d}{dt}]\right)^{n\times n}$ and $V \in \left(\boldsymbol{R}[\frac{d}{dt}]\right)^{(n+m)\times(n+m)}$ satisfying (2.22). Although in general, it has been only known that algebraic controllability of nonlinear systems (2.1)-(2.2) is a necessary condition for differential flatness [66], in the case of linear time invariant systems (2.20)-(2.21), the concept of algebraic controllability coincides with the concept of linear flatness.

**Example 2.1** Let us consider a simple system

$$\begin{cases} \dot{x}_1 = x_1 u, \\ \dot{x}_2 = x_1. \end{cases} \tag{2.27}$$

Differentiating both sides of system (2.27), we have $\underbrace{\begin{pmatrix} \frac{d}{dt} - u & 0 & -x_1 \\ -1 & \frac{d}{dt} & 0 \end{pmatrix}}_{P^c_{(x_1,x_2,u)}} \begin{pmatrix} dx_1 \\ dx_2 \\ du \end{pmatrix} = 0$. By elementary row and column operations, we have

$$\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}_{U^c_{(x_1,x_2,u)}} P^c_{(x_1,x_2,u)} \underbrace{\begin{pmatrix} 0 & 1 & \frac{d}{dt} \\ 0 & 0 & 1 \\ -\frac{1}{x_1} & \frac{1}{x_1}\frac{d}{dt} - \frac{u}{x_1} & \frac{1}{x_1}\frac{d^2}{dt^2} - \frac{u}{x_1}\frac{d}{dt} \end{pmatrix}}_{V^c_{(x_1,x_2,u)}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Hence system (2.27) is algebraically controllable.

Furthermore,

$$\begin{aligned} \left(U^c_{(x_1,x_2,u)}\right)^{-1} &:= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \\ \left(V^c_{(x_1,x_2,u)}\right)^{-1} &:= \begin{pmatrix} \frac{d}{dt} - u & 0 & -x_1 \\ 1 & -\frac{d}{dt} & 0 \\ 0 & 1 & 0 \end{pmatrix}. \end{aligned}$$

Viewing each element of $P^c_{(x_1,x_2,u)}$, $U^c_{(x_1,x_2,u)}$, $(U^c_{(x_1,x_2,u)})^{-1}$, $V^c_{(x_1,x_2,u)}$, $(V^c_{(x_1,x_2,u)})^{-1}$, we can know that any (periodic) trajectory $(x_1^*(t), x_2^*(t), u^*(t)) \in (C^\infty_{\mathrm{pw}})^3$ such that $x_1^*(t)$ and $u^*(t)$ are bounded on $\boldsymbol{R}$, and $x_1^*(t) \neq 0$ for almost all $t \in \boldsymbol{R}$ is a (periodic) controllable trajectory.

Since

$$\begin{cases} x_1 = \dot{x}_2, \\ u = \frac{\ddot{x}_2}{\dot{x}_2}, \end{cases} \tag{2.28}$$

system (2.27) is differentially flat with a flat output $x_2$. Thus we can easily find a controllable trajectory as follows.

$$x_1^*(t) = \begin{cases} 1 + \exp(-\frac{1}{t}) + \frac{1}{t}\exp(-\frac{1}{t}) & (t > 0), \\ 1 & (t \leq 0), \end{cases}$$

$$x_2^*(t) = \begin{cases} t + t\exp(-\frac{1}{t}) & (t > 0), \\ t & (t \leq 0), \end{cases}$$

$$u^*(t) = \begin{cases} \frac{\exp(-\frac{1}{t})}{t^3(1+\exp(-\frac{1}{t})+t^2\exp(-\frac{1}{t})} & (t > 0), \\ 0 & (t \leq 0). \end{cases}$$

On the other hand, although $(\tilde{x}_1^*(t), \tilde{x}_2^*(t), \tilde{u}^*(t)) = (0, 0, 1)$ is a smooth trajectory on $\boldsymbol{R}$ of system (2.27), $V^c_{(\tilde{x}_1^*(t),\tilde{x}_2^*(t),\tilde{u}^*(t))} \notin \left(C^\infty_{\text{a.e.}}[\frac{d}{dt}]\right)^{(n+m)\times(n+m)}$. Therefore at this stage, we cannot conclude that $(\tilde{x}_1^*(t), \tilde{x}_2^*(t), \tilde{u}^*(t))$ is a controllable trajectory. ■

**Example 2.2** Let us consider a nonholonomic mobile robot [5, 39, 44, 51–53, 62, 76]

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix} u_1 + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u_2, \tag{2.29}$$

where $(x, y)$ and $\theta$ denote the wheel-axis-center position and the orientation of the robot, respectively, and $u_1$ and $u_2$ denote the translational and rotational velocities, respectively. Here $(x, y, \theta)$ and $(u_1, u_2)$ denote state and input variables, respectively. Differentiating both sides of system (2.29), we have

$$\underbrace{\begin{pmatrix} \frac{d}{dt} & 0 & \sin\theta u_1 & -\cos\theta & 0 \\ 0 & \frac{d}{dt} & -\cos\theta u_1 & -\sin\theta & 0 \\ 0 & 0 & \frac{d}{dt} & 0 & -1 \end{pmatrix}}_{P^c_{(x,y,\theta,u_1,u_2)}} \begin{pmatrix} dx \\ dy \\ d\theta \\ du_1 \\ du_2 \end{pmatrix} = 0.$$

By elementary column and row operations, we have

$$U_{(x,y,\theta,u_1,u_2)} P^c_{(x,y,\theta,u_1,u_2)} V_{(x,y,\theta,u_1,u_2)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \tag{2.30}$$

where

$$U_{(x,y,\theta,u_1,u_2)} := \begin{pmatrix} 1 & -\frac{\cos\theta}{\sin\theta} & 0 \\ 0 & -\frac{1}{\sin\theta} & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

$$V_{(x,y,\theta,u_1,u_2)} := \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & \sin^2\theta & 0 \\ \frac{\sin\theta}{u_1} & 0 & 0 & \frac{\sin^2\theta\cos\theta}{u_1}\frac{d}{dt} + 2\sin\theta\cos^2\theta\frac{u_2}{u_1} & -\frac{\sin\theta}{u_1}\frac{d}{dt} \\ -\cos\theta & 1 & 0 & \sin^3\theta\frac{d}{dt} + 2\sin^2\theta\cos\theta u_2 & \cos\theta\frac{d}{dt} \\ \alpha & 0 & 1 & \alpha\sin\theta\cos\theta\frac{d}{dt} + 2\alpha\cos^2\theta u_2 & -\alpha\frac{d}{dt} \end{pmatrix},$$

and where

$$\alpha := \frac{\sin\theta}{u_1}\frac{d}{dt} + \cos\theta\frac{u_2}{u_1} - \frac{\dot{u}_1}{u_1^2}.$$

Hence system (2.29) is algebraically controllable.

Furthermore,

$$U^{-1}_{(x,y,\theta,u_1,u_2)} = \begin{pmatrix} 1 & -\cos\theta & 0 \\ 0 & -\sin\theta & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

$$V^{-1}_{(x,y,\theta,u_1,u_2)} = \begin{pmatrix} \frac{d}{dt} & -\frac{\cos\theta}{\sin\theta}\frac{d}{dt} & \frac{u_1}{\sin\theta} & 0 & 0 \\ 0 & -\frac{1}{\sin\theta}\frac{d}{dt} & \frac{\cos\theta}{\sin\theta}u_1 & 1 & 0 \\ 0 & 0 & -\frac{d}{dt} & 0 & 1 \\ 0 & \frac{1}{\sin^2\theta} & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Viewing each element of $P^c_{(x,y,\theta,u_1,u_2)}$, $U_{(x,y,\theta,u_1,u_2)}$, $V_{(x,y,\theta,u_1,u_2)}$, $(U_{(x,y,\theta,u_1,u_2)})^{-1}$, $(V_{(x,y,\theta,u_1,u_2)})^{-1}$, we can know that any (periodic) trajectory $(x^*(t), y^*(t), \theta^*(t), u_1^*(t), u_2^*(t)) \in (C^\infty_{\text{pw}})^5$ of system (2.29) such that $u_1^*(t)$ is bounded on $\boldsymbol{R}$, and $u_1^*(t) \neq 0$ and $\theta(t)^* \neq n\pi$ for almost all $t \in \boldsymbol{R}$, $n \in \boldsymbol{Z}$, is a (periodic) controllable trajectory. ■

Let $(x^*(t), u^*(t)) \in (C^\infty_{\text{pw}})^n \times (C^\infty_{\text{pw}})^m$ be any trajectory of system (2.1)-(2.2) such that $P^c_{(x^*(t),u^*(t))}$ is bounded. Then we can define the behavior

$$\mathcal{B}_{(x^*(t),u^*(t))} := \left\{ w \in (C^\infty_{\text{a.e.}})^{n+m} \;\middle|\; P^c_{(x^*(t),u^*(t))} w = 0 \right\}. \tag{2.31}$$

Controllability of the behavior $\mathcal{B}_{(x^*(t),u^*(t))}$ is defined in the same way as definition B.6 in appendix B. In general, we cannot guarantee that there exist $U_t$, $U_t^{-1} \in \left(C^\infty_{\text{a.e.}}[\frac{d}{dt}]\right)^{n\times n}$ and $V_t$, $V_t^{-1} \in \left(C^\infty_{\text{a.e.}}[\frac{d}{dt}]\right)^{(n+m)\times(n+m)}$ such that

$$U_t P^c_{(x^*(t),u^*(t))} V_t = \begin{pmatrix} I_n & 0 \end{pmatrix}.$$

However, if system (2.1)-(2.2) is algebraically controllable, there exist unimodular matrices $U_{(x,u)} \in \mathcal{D}_{(x,u)}^{n\times n}$ and $V_{(x,u)} \in \mathcal{D}_{(x,u)}^{(n+m)\times(n+m)}$ satisfying (2.16). Thus if $(x^*(t), u^*(t))$ is a controllable trajectory,

$$U_{(x^*(t),u^*(t))}P^c_{(x^*(t),u^*(t))}V_{(x^*(t),u^*(t))} = \begin{pmatrix} I_n & 0 \end{pmatrix},$$

where $U_{(x^*(t),u^*(t))}$, $U^{-1}_{(x^*(t),u^*(t))} \in \left(C^\infty_{\text{a.e.}}[\frac{d}{dt}]\right)^{n\times n}$ and $V_{(x^*(t),u^*(t))}$, $V^{-1}_{(x^*(t),u^*(t))} \in \left(C^\infty_{\text{a.e.}}[\frac{d}{dt}]\right)^{(n+m)\times(n+m)}$. From this, we have the following lemma.

**Lemma 2.16** *Suppose that system* (2.1)-(2.2) *is algebraically controllable. Let* $(x^*(t), u^*(t))$ *be any controllable trajectory. Then behavior* $\mathcal{B}_{(x^*(t),u^*(t))}$ *is controllable.*

**Proof** Since system (2.1)-(2.2) is algebraically controllable, there exist unimodular matrices $U_{(x,u)} \in \mathcal{D}_{(x,u)}^{n\times n}$ and $V_{(x,u)} \in \mathcal{D}_{(x,u)}^{(n+m)\times(n+m)}$ satisfying (2.16). Since $(x^*(t), u^*(t))$ is a controllable trajectory, we have

$$\begin{aligned}\mathcal{B}_{(x^*(t),u^*(t))} &= \left\{w \in (C^\infty_{\text{a.e.}})^{n+m} \;\middle|\; U_{(x^*(t),u^*(t))}P^c_{(x^*(t),u^*(t))}V_{(x^*(t),u^*(t))}V^{-1}_{(x^*(t),u^*(t))}w = 0\right\} \\ &= \left\{w \in (C^\infty_{\text{a.e.}})^{n+m} \;\middle|\; W^1_t w = 0\right\},\end{aligned}$$

where

$$V^{-1}_{(x^*(t),u^*(t))} =: \begin{pmatrix} W^1_t \\ W^2_t \end{pmatrix}.$$

Note that by condition 3 of definition 2.15, $W^1_t \in \left(C^\infty_{\text{a.e.}}\left[\frac{d}{dt}\right]\right)^{n\times(n+m)}$ and $W^2_t \in \left(C^\infty_{\text{a.e.}}\left[\frac{d}{dt}\right]\right)^{m\times(n+m)}$.

Let

$$V_{(x^*(t),u^*(t))} =: \begin{pmatrix} V^1_t & V^2_t \end{pmatrix}.$$

Note that by condition 3 of definition 2.15, $V^1_t \in \left(C^\infty_{\text{a.e.}}\left[\frac{d}{dt}\right]\right)^{(n+m)\times n}$ and $V^2_t \in \left(C^\infty_{\text{a.e.}}\left[\frac{d}{dt}\right]\right)^{(n+m)\times m}$. Then $V^1_tW^1_t + V^2_tW^2_t = I_{n+m}$. Thus if $\tilde{w} \in \mathcal{B}_{(x^*(t),u^*(t))}$, we have

$$\tilde{w} = V^2_tW^2_t\tilde{w}.$$

Hence for any $w_1$, $w_2 \in \mathcal{B}_{(x^*(t),u^*(t))}$, there exist $l_1$, $l_2 \in (C^\infty_{\text{a.e.}})^m$ such that

$$\begin{cases} w_1 = V^2_t l_1, \\ w_2 = V^2_t l_2. \end{cases}$$

Let $t_0$ be in $\boldsymbol{R}\backslash(\boldsymbol{E}(l_1) \cup \boldsymbol{E}(l_2) \cup \boldsymbol{E}(V_t^2))$. Then there exists an open interval $t_0 \in \mathcal{I} \subset \boldsymbol{R}$ such that $l_1$, $l_2$ and $w_1$, $w_2$ are smooth on $\mathcal{I}$. Let $l$ be a smooth function on $\mathcal{I}$ with

$$l(t) = \begin{cases} l_1(t), & t \leq t_0, \\ l_2(t), & t \geq t_1, \end{cases}$$

where $t_1 \in \mathcal{I}$ and $t_1 > t_0$. Then we can conclude that

$$w := V_t^2 l \in \mathcal{B}_{(x^*(t),u^*(t))}.$$

In fact, since

$$V_{(x^*(t),u^*(t))}^{-1} V_{(x^*(t),u^*(t))} = \begin{pmatrix} W_t^1 V_t^1 & W_t^1 V_t^2 \\ W_t^2 V_t^1 & W_t^2 V_t^2 \end{pmatrix} = \begin{pmatrix} I_n & 0 \\ 0 & I_m \end{pmatrix},$$

we have

$$W_t^1 V_t^2 = 0.$$

Therefore

$$W_t^1 w = W_t^1 (V_t^2 l) = 0.$$

Hence for any $w_1$, $w_2 \in \mathcal{B}_{(x^*(t),u^*(t))}$ and almost all $t_0 \in \boldsymbol{R}$, there exist $w \in \mathcal{B}_{(x^*(t),u^*(t))}$, an open interval $t_0 \in \mathcal{I} \subset \boldsymbol{R}$, and $t_1 > t_0$ such that $w_1$, $w_2$, $w$ are smooth on $\mathcal{I}$ and for all $t \in \mathcal{I}$

$$w(t) = \begin{cases} w_1(t), & t \leq t_0, \\ w_2(t), & t \geq t_1. \end{cases}$$

Therefore $\mathcal{B}_{(x^*(t),u^*(t))}$ is controllable. □

By lemma 2.16, if $(x^*(t), u^*(t))$ is a controllable trajectory, the behavior $\mathcal{B}_{(x^*(t),u^*(t))}$ defined by (2.31) is controllable. Since

$$\mathcal{B}_{(x^*(t),u^*(t))} = \left\{ (x_\epsilon, u_\epsilon) \in (C^\infty_{\text{a.e.}})^n \times (C^\infty_{\text{a.e.}})^m \;\middle|\; \dot{x}_\epsilon = A(t)x_\epsilon + B(t)u_\epsilon \right\},$$

for all $(x_{\epsilon,0}, u_{\epsilon,0})$, $(x_{\epsilon,1}, u_{\epsilon,1}) \in \mathcal{B}_{(x^*(t),u^*(t))}$, and for almost all $t_0 \in \boldsymbol{R}$, there exist $(x_\epsilon, u_\epsilon) \in \mathcal{B}_{(x^*(t),u^*(t))}$, an open interval $t_0 \in \mathcal{I} \subset \boldsymbol{R}$ and $t_1 > t_0$ with $t_1 \in \mathcal{I}$ such that $(x_{\epsilon,0}, u_{\epsilon,0})$, $(x_{\epsilon,1}, u_{\epsilon,1})$, $(x_\epsilon, u_\epsilon)$ are smooth on $\mathcal{I}$ and

$$(x_\epsilon(t), u_\epsilon(t)) = \begin{cases} (x_{\epsilon,0}(t), u_{\epsilon,0}(t)), & \text{if } t \leq t_0, \\ (x_{\epsilon,1}(t), u_{\epsilon,1}(t)), & \text{if } t \geq t_1, \end{cases} \tag{2.32}$$

where $A(\cdot)$ and $B(\cdot)$ are defined in (2.4). Since we can take any $x_\epsilon(t_0) \in \boldsymbol{R}^n$ and any $x_\epsilon(t_1) \in \boldsymbol{R}^n$, the relation (2.32) implies that for all $e \in \boldsymbol{R}^n$ and almost all $t_0 \in \boldsymbol{R}$, there exist an open interval $t_0 \in \mathcal{I}$ and $t_1 > t_0$ with $t_1 \in \mathcal{I}$, and a control input $u_\epsilon \in C^\infty(\mathcal{I}, \boldsymbol{R}^m)$ such that $x_\epsilon(t_0) = e$ and $x_\epsilon(t_1) = 0$. Actually, this is satisfied at all $t_0 \in \boldsymbol{R}$. Namely, we have the following theorem.

**Theorem 2.6** *Suppose that system* (2.1)-(2.2) *is algebraically controllable. Then every linearized system* (2.4)-(2.5) *along any controllable trajectory* $(x^*(t), u^*(t))$ *of system* (2.1)-(2.2) *is completely controllable.*

**Proof** Suppose that there exists a singular point $t_s \in \boldsymbol{R}$, that is, for all $e \in \boldsymbol{R}^n$, there do not exist $t_1 > t_s$ and a control input $u_\epsilon \in C^\infty(\mathcal{I}, \boldsymbol{R}^m)$ such that $x_\epsilon(t_s) = e$ and $x_\epsilon(t_1) = 0$. Since $(x^*(t), u^*(t))$ is a controllable trajectory of system (2.1)-(2.2), $A(\cdot)$ and $B(\cdot)$ are bounded on $\boldsymbol{R}$. Thus if we apply $u_\epsilon = 0$ into system (2.4), for all $t_0 \in \boldsymbol{R}$ and $e \in \boldsymbol{R}^n$, we have an absolutely continuous solution $x_\epsilon(t)$ on $\boldsymbol{R}$ such that $x_\epsilon(t_0) = e$ (see appendix C in [102]). Hence if we apply $u_\epsilon = 0$ into system (2.4), for all $\delta > 0$, there exists $\tilde{e} \in \boldsymbol{R}^n$ such that $x_\epsilon(t_s + \delta) = \tilde{e}$. Note that if we take $\delta$ sufficiently small, $t_s + \delta$ is not a singular point. Therefore for all $e \in \boldsymbol{R}^n$, there exist an open interval $t_s \in \mathcal{I}_s$, $t_1 > t_s + \delta$ with $t_1 \in \mathcal{I}_s$ and a control input $u_\epsilon \in C^\infty(\mathcal{I}_s, \boldsymbol{R}^m)$ such that $x_\epsilon(t_s) = e$ and $x_\epsilon(t_1) = 0$. This is a contradiction because $t_s$ is a singular point. Hence there does not exist any singular point. Thus for all $e \in \boldsymbol{R}^n$ and all $t_0 \in \boldsymbol{R}$, there exist an open interval $t_0 \in \mathcal{I}$, $t_1 > t_0$ with $t_1 \in \mathcal{I}$, and a control input $u_\epsilon \in C^\infty(\mathcal{I}, \boldsymbol{R}^m)$ such that $x_\epsilon(t_0) = e$ and $x_\epsilon(t_1) = 0$. This means that linear system (2.4)-(2.5) is completely controllable. □

By applying theorem 2.6 and a result of [101], we can relate algebraic controllability and uniform complete controllability.

**Corollary 2.7** *Suppose that system* (2.1)-(2.2) *is algebraically controllable. Then every linearized system* (2.4)-(2.5) *along any periodic controllable trajectory* $(x^*(t), u^*(t))$ *of system* (2.1)-(2.2) *is uniformly completely controllable.*

**Proof** Since $(x^*(t), u^*(t))$ is a controllable trajectory, the matrix $P^c_{(x^*(t),u^*(t))}$ is bounded. Hence $A(t)$ and $B(t)$ of system (2.4) are bounded on $\boldsymbol{R}$ because $P^c_{(x^*(t),u^*(t))} \begin{pmatrix} x_\epsilon \\ u_\epsilon \end{pmatrix} = 0$ is equivalent to system (2.4). Moreover, if a controllable trajectory $(x^*(t), u^*(t))$ is periodic, $A(t)$ and $B(t)$ are periodic. Then system (2.4)-(2.5) is uniformly completely controllable if and only if system (2.4)-(2.5) is completely controllable [101]. Hence by theorem 2.6, we obtain the conclusion. □

By corollary 2.7, answers in questions 2.1 and 2.2 are algebraically controllable and periodic controllable trajectory, respectively.

**Interpretation 2.1** *Suppose that a given system* (2.1)-(2.2) *is algebraically controllable and* $(x^*(t), u^*(t))$ *is a periodic controllable trajectory. Let* $x^*(t)$ *be the reference trajectory of system* (2.1)-(2.2)*. Then by corollary 2.7, as mentioned in section 2.1, there exist a feedback gain* $K(t)$ *such that if we apply* (2.8) *into system* (2.1)-(2.2)*, the actual trajectory* $x(t)$ *locally exponentially approaches the reference trajectory* $x^*(t)$*.*

As shown in example 2.2, any trajectory $(x^*(t), y^*(t), \theta^*(t), u_1^*(t), u_2^*(t)) \in (C_{\rm pw}^\infty)^5$ of system (2.29) such that $u_1^*(t)$ is bounded on $\boldsymbol{R}$, and $u_1^*(t) \neq 0$ and $\theta^*(t) \neq n\pi$ for almost all $t \in \boldsymbol{R}$, $n \in \boldsymbol{Z}$, is a controllable trajectory of system (2.29). Hence by theorem 2.6 and corollary 2.7, a linearized system along such a (periodic) trajectory in the class

$$\begin{pmatrix} \dot{x}_\epsilon \\ \dot{y}_\epsilon \\ \dot{\theta}_\epsilon \end{pmatrix} = \begin{pmatrix} 0 & 0 & -\sin(\theta^*(t))u_1^*(t) \\ 0 & 0 & \cos(\theta^*(t))u_1^*(t) \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_\epsilon \\ y_\epsilon \\ \theta_\epsilon \end{pmatrix} + \begin{pmatrix} \cos(\theta^*(t)) & 0 \\ \sin(\theta^*(t)) & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u_{1,\epsilon} \\ u_{2,\epsilon} \end{pmatrix}$$

is (uniformly) completely controllable. However, it has not been clarified that a trajectory $(x^*(t), y^*(t), 0, u_1^*(t), u_2^*(t))$ of system (2.29) is a controllable trajectory of system (2.29), yet. The next subsection explains that if a trajectory of system (2.1)-(2.2) is **analytic** on $\boldsymbol{R}$, we can apply a result of appendix B.

### 2.2.1 Analytic trajectory

This subsection explains that if a trajectory of system (2.1)-(2.2) is analytic on $\boldsymbol{R}$, we can examine complete controllability of a linearized system along the analytic trajectory by using a result of appendix B. Let $\mathcal{D}_t := \mathcal{M}_t[\frac{d}{dt}]$, where $\mathcal{M}_t$ is a field of all meromorphic functions on $\boldsymbol{R}$ with respect to $t$ (see appendix B.3 and appendix D).

First, we prepare the following lemma.

**Lemma 2.17** *Suppose that a trajectory $(x^*(t), u^*(t))$ of system (2.1)-(2.2) is analytic on $\boldsymbol{R}$ such that $P^c_{(x^*(t),u^*(t))}$ is bounded. Then $P^c_{(x^*(t),u^*(t))} \in (\mathcal{D}_t)^{n\times(n+m)}$*

**Proof** Since every coefficient function of each polynomial element of $P^c_{(x,u)}$ is a meromorphic function depending on a finite number of variables of $\{x, u, \dot{u}, \cdots\}$, we can describe it as $\frac{\beta(x,u,\dot{u},\cdots)}{\alpha(x,u,\dot{u},\cdots)}$ by using some analytic functions $\alpha$ and $\beta$. Since a composite function of an analytic function and an analytic function is analytic [57], substituting $(x^*(t), u^*(t))$ into $(x, u)$ of $\alpha(x, u, \dot{u}, \cdots)$ and $\beta(x, u, \dot{u}, \cdots)$, $\tilde{\alpha}(t) := \alpha(x^*(t), u^*(t), \dot{u}^*(t), \cdots)$ and $\tilde{\beta}(t) := \beta(x^*(t), u^*(t), \dot{u}^*(t), \cdots)$ are analytic on $\boldsymbol{R}$. In addition, since $P^c_{(x^*(t),u^*(t))}$ is bounded, we have $\tilde{\alpha}(t) \neq 0$ on $\boldsymbol{R}$. Hence every coefficient function of each polynomial element of $P^c_{(x^*(t),u^*(t))}$ can be described as the form $\frac{\tilde{\beta}(t)}{\tilde{\alpha}(t)}$ which is a meromorphic function with respect to $t$. □

Let $(x^*(t), u^*(t))$ be an **analytic** trajectory on $\boldsymbol{R}$ of system (2.1)-(2.2) such that $P^c_{(x^*(t),u^*(t))}$ is bounded. Thus by lemma 2.17 and proposition B.8 in appendix B, the behavior $\mathcal{B}_{(x^*(t),u^*(t))}$ defined by (2.31) is controllable if and only if there exist unimodular matrices $U_t \in \mathcal{D}_t^{n\times n}$ and $V_t \in \mathcal{D}_t^{(n+m)\times(n+m)}$ such that

$$U_t P^c_{(x^*(t),u^*(t))} V_t = \begin{pmatrix} I_n & 0 \end{pmatrix}. \tag{2.33}$$

Hence then in the same way as the proof of theorem 2.6 and corollary 2.7, we get the following corollary.

**Corollary 2.8** *Suppose that a (periodic) trajectory* $(x^*(t), u^*(t))$ *of system* (2.1)-(2.2) *is analytic on* $\boldsymbol{R}$ *such that* $P^c_{(x^*(t),u^*(t))}$ *is bounded. Then if there exist unimodular matrices* $U_t \in \mathcal{D}_t^{n\times n}$ *and* $V_t \in \mathcal{D}_t^{(n+m)\times(n+m)}$ *satisfying* (2.33)*, a linearized system* (2.4)-(2.5) *along the trajectory* $(x^*(t), u^*(t))$ *is (uniformly) completely controllable.*

**Example 2.3** Let us go back to example 2.2, again. Suppose that $(x^*(t), y^*(t), 0, u_1^*(t), u_2^*(t))$ of system (2.29) is an analytic trajectory. Then we have

$$P_t^c := P^c_{(x^*(t),y^*(t),0,u_1^*(t),u_2^*(t))} = \begin{pmatrix} \frac{d}{dt} & 0 & u_1^*(t) & -1 & 0 \\ 0 & \frac{d}{dt} & -u_1^*(t) & 0 & 0 \\ 0 & 0 & \frac{d}{dt} & 0 & -1 \end{pmatrix}.$$

By elementary row and column operations, we have

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} P_t^c \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & -\frac{1}{u_1^*(t)} & 0 & 0 & \frac{1}{u_1^*(t)}\frac{d}{dt} \\ 1 & 0 & 0 & \frac{d}{dt} & 0 \\ 0 & \alpha & 1 & 0 & -\alpha\frac{d}{dt} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix},$$

where $\alpha := -\frac{1}{u_1^*(t)}\frac{d}{dt} + \frac{\dot{u}_1^*(t)}{(u_1^*(t))^2}$. Hence by corollary 2.8, a linearized system along the trajectory $(x^*(t), y^*(t), 0, u_1^*(t), u_2^*(t))$

$$\begin{pmatrix} \dot{x}_\epsilon \\ \dot{y}_\epsilon \\ \dot{\theta}_\epsilon \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & u_1^*(t) \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_\epsilon \\ y_\epsilon \\ \theta_\epsilon \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u_{1,\epsilon} \\ u_{2,\epsilon} \end{pmatrix}$$

is completely controllable, where $u_1^*(t)$ is bounded on $\boldsymbol{R}$ and $u_1^*(t) \neq 0$ for almost all $t \in \boldsymbol{R}$. ■

**Remark 2.8** *If a trajectory* $(x^*(t), u^*(t))$ *is analytic and* $P^c_{(x^*(t),u^*(t))}$ *is bounded, we can use corollary 2.8 without calculating* $U_t^{-1}$ *and* $V_t^{-1}$*. Note that if a trajectory* $(x^*(t), u^*(t)) \in (C^\infty_{\mathrm{pw}})^n \times (C^\infty_{\mathrm{pw}})^m$ *is not analytic, to use theorem 2.6 and corollary 2.7, we have to obtain* $U^{-1}_{(x,u)}$ *and* $V^{-1}_{(x,u)}$*, where* $U_{(x,u)}$ *and* $V_{(x,u)}$ *are unimodular matrices satisfying* (2.16)*.* ■

## 2.3 Algebraic observability

In order to answer the questions 2.3 and 2.4, this section introduces novel concepts called algebraic observability and observable trajectory of system (2.1)-(2.2). It is shown that if a given nonlinear system (2.1)-(2.2) is algebraically observable, then every linearized system along any (periodic) observable trajectory is (uniformly) completely observable.

Differentiating both sides of Eqs. (2.1)-(2.2), we have

$$P^o_{(x,,u)}dx = Q^o_{(x,u)}\begin{pmatrix} du \\ dy \end{pmatrix}, \tag{2.34}$$

where

$$\begin{aligned} P^o_{(x,u)} &:= \begin{pmatrix} \frac{d}{dt}I_n - \frac{\partial f}{\partial x}(x,u) \\ -\frac{\partial h}{\partial x}(x) \end{pmatrix} \in \mathcal{D}^{(n+p)\times n}_{(x,u)}, \\ Q^o_{(x,u)} &:= \begin{pmatrix} \frac{\partial f}{\partial u}(x,u) & 0 \\ 0 & -I_p \end{pmatrix} \in \mathcal{D}^{(n+p)\times(m+p)}_{(x,u)}. \end{aligned} \tag{2.35}$$

In the same way as algebraic controllability, algebraic observability is defined.

**Definition 2.18** *System* (2.1)-(2.2) *is called* **algebraically observable** *if* $P^o_{(x,u)}$ *defined by* (2.35) *is hyper-regular, that is, there exist unimodular matrices* $U_{(x,u)} \in \mathcal{D}^{(n+p)\times(n+p)}_{(x,u)}$ *and* $V_{(x,u)} \in \mathcal{D}^{n\times n}_{(x,u)}$ *such that*

$$U_{(x,u)}P^o_{(x,u)}V_{(x,u)} = \begin{pmatrix} I_n \\ 0_{p\times n} \end{pmatrix}. \tag{2.36}$$

Differentiating both sides of (2.20)-(2.21), we have

$$\underbrace{\begin{pmatrix} \frac{d}{dt}I_n - A \\ -C \end{pmatrix}}_{P^o_{\text{linear}}} dx = \begin{pmatrix} B & 0 \\ 0 & -I_p \end{pmatrix}\begin{pmatrix} du \\ dy \end{pmatrix}.$$

We note that $P^o_{\text{linear}} \in \left(\boldsymbol{R}[\frac{d}{dt}]\right)^{(n+p)\times n} \subset \mathcal{D}^{(n+p)\times n}_{(x,u)}$. Since $P^o_{\text{linear}}$ can be transformed into the Smith form, we should say that system (2.20)-(2.21) is algebraically observable if there exist unimodular matrices $U \in \left(\boldsymbol{R}[\frac{d}{dt}]\right)^{(n+p)\times(n+p)}$ and $V \in \left(\boldsymbol{R}[\frac{d}{dt}]\right)^{n\times n}$ such that

$$UP^o_{\text{linear}}V = \begin{pmatrix} I_n \\ 0 \end{pmatrix}. \tag{2.37}$$

It is known [88] that system (2.20)-(2.21) is observable if and only if there exist unimodular matrices $U \in \left(\boldsymbol{R}[\frac{d}{dt}]\right)^{n\times n}$ and $V \in \left(\boldsymbol{R}[\frac{d}{dt}]\right)^{(n+m)\times(n+m)}$ satisfying

(2.37). Therefore system (2.20)-(2.21) is algebraically observable if and only if system (2.20)-(2.21) is observable.

Similarly to the case of algebraic controllability, algebraic observability is invariant under an analytic coordinate transformation.

**Theorem 2.9** *Suppose that system* (2.1)-(2.2) *is algebraically observable. Moreover, suppose that* $x = \phi(\hat{x})$ *is an analytic coordinate transformation with the analytic inverse. Then the transformed system* (2.23)-(2.24) *is also algebraically observable.*

**Proof** Differentiating both sides of (2.23)-(2.24), we have

$$\underbrace{\begin{pmatrix} \frac{d}{dt}I - \frac{\partial}{\partial \hat{x}}\left[A(\hat{x})^{-1}f(\phi(\hat{x}),u)\right] \\ -\frac{\partial h}{\partial x}(\phi(\hat{x}))A(\hat{x}) \end{pmatrix}}_{\hat{P}^o_{(\hat{x},u)}} d\hat{x} = \begin{pmatrix} A(\hat{x})^{-1}\frac{\partial f}{\partial u}(\phi(\hat{x}),u) & 0 \\ 0 & -I \end{pmatrix} \begin{pmatrix} du \\ dy \end{pmatrix},$$

where $A(\hat{x}) := \frac{\partial \phi}{\partial \hat{x}}(\phi(\hat{x}))$. In the same way as the proof of theorem 2.2, we obtain

$$\hat{P}^o_{(\hat{x},u)} = \begin{pmatrix} A(\hat{x})^{-1} & 0 \\ 0 & I \end{pmatrix} P^o_{(\phi(\hat{x}),u)} A(\hat{x}).$$

On the other hand, system (2.1)-(2.2) is algebraically observable, there exist unimodular matrices $U_{(x,u)} \in \mathcal{D}^{(n+p)\times(n+p)}_{(x,u)}$ and $V_{(x,u)} \in \mathcal{D}^{n\times n}_{(x,u)}$ satisfying (2.36). Hence we have

$$\underbrace{\left[U_{(\phi(\hat{x}),u)} \begin{pmatrix} A(\hat{x}) & 0 \\ 0 & I \end{pmatrix}\right]}_{\hat{U}_{(\hat{x},u)}} \hat{P}_{(\hat{x},u)} \underbrace{\left[A(\hat{x})^{-1} V_{(\phi(\hat{x}),u)}\right]}_{\hat{V}_{(\hat{x},u)}} = \begin{pmatrix} I_n & 0 \end{pmatrix}.$$

Since $\hat{U}_{(\hat{x},u)}$ and $\hat{V}_{(\hat{x},u)}$ are unimodular, system (2.23)-(2.24) is also algebraically observable. □

As a corollary of theorem 2.9, algebraic observability is invariant under a linear coordinate transformation.

**Corollary 2.10** *Suppose that system* (2.1)-(2.2) *is algebraically observable. Let* $x = A\tilde{x}$*, where* $A \in \boldsymbol{R}^{n\times n}$ *is invertible. Then the transformed system* (2.23)-(2.24) *is also algebraically observable.*

Similarly to the case of algebraic controllability, in order to relate algebraic observability and uniform complete observability, we define observable trajectory of system (2.1)-(2.2).

**Definition 2.19** *Suppose that system* (2.1)-(2.2) *is algebraically observable. A trajectory* $(x^*(t), u^*(t))$ *of system* (2.1)-(2.2) *is called an* **observable trajectory** *if the following conditions are satisfied:*

1. $(x^*, u^*) \in (C^\infty_{\text{pw}})^n \times (C^\infty_{\text{pw}})^m$.
2. *The matrices* $P^o_{(x^*(t),u^*(t))}$ *and* $Q^o_{(x^*(t),u^*(t))}$ *are bounded.*
3. *There exist unimodular matrices* $U_{(x,u)} \in \mathcal{D}^{(n+p)\times(n+p)}_{(x,u)}$ *and* $V_{(x,u)} \in \mathcal{D}^{n\times n}_{(x,u)}$ *satisfying* (2.36) *such that*

$$U_{(x^*(t),u^*(t))}, U^{-1}_{(x^*(t),u^*(t))} \in \left(C^\infty_{\text{a.e.}}\left[\frac{d}{dt}\right]\right)^{(n+p)\times(n+p)},$$
$$V_{(x^*(t),u^*(t))}, V^{-1}_{(x^*(t),u^*(t))} \in \left(C^\infty_{\text{a.e.}}\left[\frac{d}{dt}\right]\right)^{n\times n}.$$

In the case of linear time invariant systems (2.20)-(2.21), the concept of observability coincides with the concept of complete observability [10]. Hence if we consider linear time invariant systems (2.20)-(2.21), in order to characterize algebraic observability, the concept of observable trajectory is not required.

**Example 2.4** Let us go back to example 2.1 with an output variable $y = x_2$. That is, let us consider

$$\begin{cases} \dot{x}_1 = x_1 u, \\ \dot{x}_2 = x_1, \\ y = x_2. \end{cases} \tag{2.38}$$

Differentiating both sides of system (2.38), we have

$$\underbrace{\begin{pmatrix} \frac{d}{dt} - u & 0 \\ -1 & \frac{d}{dt} \\ -1 & 0 \end{pmatrix}}_{P^o_{(x_1,x_2,u)}} \begin{pmatrix} dx_1 \\ dx_2 \end{pmatrix} = \underbrace{\begin{pmatrix} x_1 & 0 \\ 0 & 0 \\ 0 & -1 \end{pmatrix}}_{Q^o_{(x_1,x_2,u)}} \begin{pmatrix} du \\ dy \end{pmatrix}$$

Repeating elementary row and column operations for $P^o_{(x,y,\theta,u_1,u_2)}$, we have

$$\underbrace{\begin{pmatrix} 0 & 1 & \frac{d}{dt} \\ 0 & 0 & 1 \\ 1 & \frac{d}{dt} - u & \frac{d^2}{dt^2} - u\frac{d}{dt} \end{pmatrix}}_{U^o_{(x_1,x_2,u)}} P^o_{(x_1,x_2,u)} \underbrace{\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}}_{V^o_{(x_1,x_2,u)}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Hence system (2.38) is algebraically observable. Furthermore, we have

$$\begin{aligned}\left(U^o_{(x_1,x_2,u)}\right)^{-1} &= \begin{pmatrix} -\frac{d}{dt}+u & 0 & 1 \\ 1 & -\frac{d}{dt} & 0 \\ 0 & 1 & 0 \end{pmatrix}, \\ \left(V^o_{(x_1,x_2,u)}\right)^{-1} &= \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.\end{aligned}$$

Viewing each element of $P^o_{(x_1,x_2,u)}$, $Q^o_{(x_1,x_2,u)}$, $U^o_{(x_1,x_2,u)}$, $V^o_{(x_1,x_2,u)}$, $(U^o_{(x_1,x_2,u)})^{-1}$, $(V^o_{(x_1,x_2,u)})^{-1}$, we can know that any (periodic) smooth trajectory $(x_1^*(t), x_2^*(t), u^*(t))$ such that $x_1^*(t)$ and $u^*(t)$ are bounded on $\boldsymbol{R}$ is a (periodic) controllable trajectory. ■

**Example 2.5** Let us go back to example 2.2 with output variables $y_1 = x$, $y_2 = y$. That is, let us consider

$$\begin{cases} \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix} u_1 + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u_2, \\ y_1 = x, \\ y_2 = y. \end{cases} \tag{2.39}$$

Differentiating both sides of (2.39), we have

$$\underbrace{\begin{pmatrix} \frac{d}{dt} & 0 & \sin\theta u_1 \\ 0 & \frac{d}{dt} & -\cos\theta u_1 \\ 0 & 0 & \frac{d}{dt} \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}}_{P^o_{(x,y,\theta,u_1,u_2)}} \begin{pmatrix} dx \\ dy \\ d\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & 0 & 0 & 0 \\ \sin\theta & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} du_1 \\ du_2 \\ dy_1 \\ dy_2 \end{pmatrix}.$$

Repeating elementary row and column operations for $P^o_{(x,y,\theta,u_1,u_2)}$, we have

$$U_{(x,y,\theta,u_1,u_2)} P^o_{(x,y,\theta,u_1,u_2)} V_{(x,y,\theta,u_1,u_2)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where

$$U_{(x,y,\theta,u_1,u_2)} := \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ \cos\theta & 0 & 0 & \cos\theta\frac{d}{dt} & 0 \\ \dot{u}_1 - u_1\frac{d}{dt} & -u_1u_2 & u_1^2\sin\theta & \dot{u}_1\frac{d}{dt} - u_1\frac{d^2}{dt^2} & -u_1u_2\frac{d}{dt} \\ \cos\theta & \sin\theta & 0 & \cos\theta\frac{d}{dt} & \sin\theta\frac{d}{dt} \end{pmatrix},$$

$$V_{(x,y,\theta,u_1,u_2)} := \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & \frac{1}{\sin\theta\cos\theta u_1} \end{pmatrix}.$$

Hence system (2.39) is algebraically observable.

Furthermore, we have

$$U^{-1}_{(x,y,\theta,u_1,u_2)} = \begin{pmatrix} -\frac{d}{dt} & 0 & \frac{1}{\cos\theta} & 0 & 0 \\ 0 & -\frac{1}{\sin\theta}\frac{d}{dt} & -\frac{1}{\sin\theta} & 0 & \frac{1}{\sin\theta} \\ 0 & 0 & \gamma & \frac{1}{u_1^2\sin\theta} & \frac{u_2}{u_1\sin^2\theta} \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix},$$

$$V^{-1}_{(x,y,\theta,u_1,u_2)} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & \sin\theta\cos\theta u_1 \end{pmatrix},$$

where $\gamma := \frac{1}{u_1\sin\theta}\left(\frac{d}{dt} - \frac{\dot{u}_1}{u_1}\right)\frac{1}{\cos\theta} - \frac{u_2}{\sin^2\theta u_1}$. Viewing each element of $P^o_{(x,y,\theta,u_1,u_2)}$, $U_{(x,y,\theta,u_1,u_2)}$, $V_{(x,y,\theta,u_1,u_2)}$, $(U_{(x,y,\theta,u_1,u_2)})^{-1}$, $(V_{(x,y,\theta,u_1,u_2)})^{-1}$, we can know that any (periodic) trajectory $(x^*(t), y^*(t), \theta^*(t), u_1^*(t), u_2^*(t)) \in (C^\infty_{\mathrm{pw}})^5$ such that $u_1^*(t)$ is bounded on $\boldsymbol{R}$, $u_1^*(t) \neq 0$ and $\theta^*(t) \neq \frac{n\pi}{2}$ for almost all $t \in \boldsymbol{R}$, $n \in \boldsymbol{Z}$, is a (periodic) observable trajectory. ∎

As a duality of theorem 2.6, we have the following theorem.

**Theorem 2.11** *Suppose that system* (2.1)-(2.2) *is algebraically observable. Then every linearized system* (2.4)-(2.5) *along any observable trajectory* $(x^*(t), u^*(t))$ *of system* (2.1)-(2.2) *is completely observable.*

**Proof** Since system (2.1)-(2.2) is algebraically observable, there exist unimodular matrices $U_{(x,u)} \in \mathcal{D}^{(n+p)\times(n+p)}_{(x,u)}$ and $V_{(x,u)} \in \mathcal{D}^{n\times n}_{(x,u)}$ satisfying (2.36). Hence we have

$$V^T_{(x,u)}(P^o_{(x,u)})^T U^T_{(x,u)} = \begin{pmatrix} I_n & 0_{n\times p} \end{pmatrix}.$$

Let $(x^*(t), u^*(t))$ be any observable trajectory of system (2.1)-(2.2) and let

$$\mathcal{B} := \left\{ w \in (C^\infty_{\mathrm{a.e.}})^{n+p} \mid \left(P^o_{(x^*(\bar{t}),u^*(\bar{t}))}\right)^T w = 0 \right\},$$

where $\bar{t} := -t$. Since $(x^*(t), u^*(t))$ is an observable trajectory, putting $\left(U_{(x^*(\bar{t}),u^*(\bar{t}))}\right)^{-1} =: \begin{pmatrix} W_t^1 \\ W_t^2 \end{pmatrix}$, where $W_t^1 \in \left(C^\infty_{\text{a.e.}}[\frac{d}{dt}]\right)^{n\times(n+p)}$ and $W_t^2 \in \left(C^\infty_{\text{a.e.}}[\frac{d}{dt}]\right)^{p\times(n+p)}$, we have

$$\begin{aligned}\mathcal{B} &= \left\{ w \in (C^\infty_{\text{a.e.}})^{n+p} \,\middle|\, V^T_{(x^*(\bar{t}),u^*(\bar{t}))} \left(P^o_{(x^*(\bar{t}),u^*(\bar{t}))}\right)^T U^T_{(x^*(\bar{t}),u^*(\bar{t}))} \left(U_{(x^*(\bar{t}),u^*(\bar{t}))}\right)^{-1} w = 0 \right\}, \\ &= \left\{ w \in (C^\infty_{\text{a.e.}})^{n+p} \,\middle|\, W_t^1 w = 0 \right\}.\end{aligned}$$

Hence, in the way as the proof of lemma 2.16, we can prove that $\mathcal{B}$ is controllable. Therefore, in the same way as the proof of theorem 2.6, we can prove that linear system

$$\frac{d\bar{x}_\epsilon}{d\bar{t}} = A(\bar{t})^T \bar{x}_\epsilon + C(\bar{t})^T \bar{u}_\epsilon, \tag{2.40}$$

is completely controllable, where $A(\cdot)$ and $C(\cdot)$ are defined in (2.4)-(2.5). Hence by the well-known duality between complete controllability and complete observability of a time varying linear system [10], complete controllability of system (2.40) is equivalent to complete observability of

$$\begin{cases} \dot{x}_\epsilon = A(t)x_\epsilon, \\ y_\epsilon = C(t)x_\epsilon. \end{cases} \tag{2.41}$$

Since $Q^o_{(x^*(t),u^*(t))}$ is bounded, complete observability of (2.41) is equivalent to that of system (2.4)-(2.5). □

Similarly to corollary 2.7, we obtain the following corollary of theorem 2.11.

**Corollary 2.12** *Suppose that system* (2.1)-(2.2) *is algebraically observable. Then every linearized system* (2.4)-(2.5) *along any periodic observable trajectory* $(x^*(t), u^*(t))$ *of system* (2.1)-(2.2) *is uniformly completely observable.*

By corollary 2.12, answers in questions 2.3 and 2.4 are algebraically observable and observable trajectory, respectively.

**Interpretation 2.2** *Suppose that an algebraically observable system was given and that we got a periodic observable trajectory* $(x^*(t), u^*(t))$ *of the system. Let* $x^*(t)$ *be the reference trajectory of system* (2.1)-(2.2)*. Then by corollary 2.12, as mentioned in section 2.1, we can design an observer gain such that the origin of error dynamics* (2.11) *between the actual error state and the estimated error state of the linearized error system* (2.4)-(2.5) *is exponentially stable. Moreover, if the system is also algebraically controllable and* $(x^*(t), u^*(t))$ *is also a controllable trajectory, as mentioned in section 2.1, it is expected that we can design a controller and an observer such that the actual trajectory* $x(t)$ *locally exponentially approaches the reference trajectory* $x^*(t)$.

Similarly to the discussion in subsection 2.2.1, in the same way as the proof of theorem 2.11, we can prove the following corollary.

**Corollary 2.13** *Suppose that a (periodic) trajectory* $(x^*(t), u^*(t))$ *of system* (2.1)-(2.2) *is analytic on* $\boldsymbol{R}$ *such that* $P^o_{(x^*(t),u^*(t))}$ *and* $Q^o_{(x^*(t),u^*(t))}$ *are bounded. Then if there exist unimodular matrices* $U_t \in \mathcal{D}_t^{(n+p)\times(n+p)}$ *and* $V_t \in \mathcal{D}_t^{n\times n}$ *satisfying*

$$U_t P^o_{(x^*(t),u^*(t))} V_t = \begin{pmatrix} I_n \\ 0 \end{pmatrix},$$

*a linearized system* (2.4)-(2.5) *along the trajectory* $(x^*(t), u^*(t))$ *is (uniformly) completely observable.*

Note that if a trajectory $(x^*(t), u^*(t))$ is analytic on $\boldsymbol{R}$, and $P^o_{(x^*(t),u^*(t))}$ and $Q^o_{(x^*(t),u^*(t))}$ are bounded, we can use corollary 2.13 without calculating $U_t^{-1}$ and $V_t^{-1}$.

**Example 2.6** Let us consider system (2.39), again. From examples 2.9 and 2.5, any (periodic) smooth trajectory $(x^*(t), y^*(t), \theta^*(t), u_1^*(t), u_2^*(t))$ such that $u_1^*(t)$ is bounded, and $u_1^*(t) \neq 0$, $\theta^*(t) \neq \frac{n\pi}{2}$ for almost all $t \in \boldsymbol{R}$, $n \in \boldsymbol{Z}$, is a (periodic) controllable and observable trajectory. By theorems 2.6, 2.11 and corollaries 2.7, 2.12, every linearized system along any (periodic) trajectories in the above class is (uniformly) completely controllable and (uniformly) completely observable. ■

The following example shows that there are non-algebraically controllable and algebraically observable systems.

**Example 2.7** Let us consider

$$\dot{x}_1 = x_1 x_2 + u, \tag{2.42}$$

$$\dot{x}_2 = -x_2, \tag{2.43}$$

$$y = x_1 \tag{2.44}$$

Differentiating both sides of (2.42)-(2.43), we have

$$\underbrace{\begin{pmatrix} \frac{d}{dt} - x_2 & -x_1 & -1 \\ 0 & \frac{d}{dt} + 1 & 0 \end{pmatrix}}_{P^c_{(x_1,x_2,u)}} \begin{pmatrix} dx_1 \\ dx_2 \\ du \end{pmatrix} = 0.$$

Repeating elementary column operations, we have

$$P^c_{(x_1,x_2,u)} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & -x_1 & \frac{d}{dt} - x_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{d}{dt} + 1 & 0 \end{pmatrix}.$$

Hence system (2.42)-(2.43) is not algebraically controllable.

On the other hand, differentiating both sides of (2.42)-(2.44), we have

$$\underbrace{\begin{pmatrix} \frac{d}{dt} - x_2 & -x_1 \\ 0 & \frac{d}{dt} + 1 \\ -1 & 0 \end{pmatrix}}_{P^o_{(x_1,x_2,u)}} \begin{pmatrix} dx_1 \\ dx_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} du \\ dy \end{pmatrix}.$$

Repeating elementary row operations, we have

$$\underbrace{\begin{pmatrix} 0 & 0 & -1 \\ -\frac{1}{x_1} & 0 & -\frac{1}{x_1}\frac{d}{dt} + \frac{x_2}{x_1} \\ \left(\frac{d}{dt} + 1\right)\frac{1}{x_1} & 1 & \left(\frac{d}{dt} + 1\right)\left(\frac{1}{x_1}\frac{d}{dt} - \frac{x_2}{x_1}\right) \end{pmatrix}}_{U_{(x_1,x_2,u)}} P^o_{(x_1,x_2,u)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Therefore system (2.42)-(2.44) is algebraically observable.

Furthermore, we have

$$U^{-1}_{(x_1,x_2,u)} = \begin{pmatrix} \frac{d}{dt} - x_2 & -x_1 & 0 \\ 0 & \frac{d}{dt} + 1 & 1 \\ -1 & 0 & 0 \end{pmatrix}.$$

Viewing each element of $P^o_{(x_1,x_2,u)}$, $U_{(x_1,x_2,u)}$, and $U^{-1}_{(x_1,x_2,u)}$, we can know that any smooth trajectory $(x_1^*(t), x_2^*(t), u^*(t))$ such that $x_1^*(t)$ and $x^2 * (t)$ are bounded on $\boldsymbol{R}$, and $x_1^*(t) \neq 0$ for almost all $t \in \boldsymbol{R}$ is an observable trajectory. Therefore by theorem 2.11 and corollary 2.12, every linearized system along any (periodic) trajectory in the above class is (uniformly) completely observable.

However, since algebraic controllability is a necessary condition for differential flatness [66], the system is not differentially flat. Thus to generate a (periodic) trajectory in the above class, we need to apply other trajectory generation techniques such as optimal control methods (see chapter 4). ■

The following example shows that there are non-algebraically controllable and non-algebraically observable systems.

**Example 2.8** Let us consider

$$\dot{x}_1 = x_1 x_2 + u, \tag{2.45}$$

$$\dot{x}_2 = -x_2, \tag{2.46}$$

$$y = x_2 \tag{2.47}$$

From example 2.7, system (2.45)-(2.46) is not algebraically controllable.

On the other hand, differentiating both sides of (2.45)-(2.47), we have

$$\underbrace{\begin{pmatrix} \frac{d}{dt} - x_2 & -x_1 \\ 0 & \frac{d}{dt} + 1 \\ 0 & -1 \end{pmatrix}}_{P^o_{(x_1,x_2,u)}} \begin{pmatrix} dx_1 \\ dx_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} du \\ dy \end{pmatrix}.$$

Repeating elementary row and column operations, we have

$$\begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & -x_1 \\ 0 & 1 & \frac{d}{dt} + 1 \end{pmatrix} P^o_{(x_1,x_2,u)} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{d}{dt} - x_2 \\ 0 & 0 \end{pmatrix}.$$

Therefore system (2.45)-(2.47) is not algebraically observable. ∎

## 2.4 Controller design for tracking of periodic reference trajectory

This section explains that if a given system is algebraically controllable and if a reference trajectory is periodic controllable trajectory, by the results of theorem 2.6 and corollary 2.7, we can design a feedback controller based on the Floquet-Lyapunov theory [9, 41, 75] and LQ optimal control theory [1]. Suppose that system (2.1)-(2.2) is algebraically controllable and there exists a controllable trajectory of the system. Then by corollary 2.7, linearizing system (2.1)-(2.2) along any controllable trajectory $(x^*(t), u^*(t))$ with period $T$, we have a uniformly completely controllable system (2.4)-(2.5), where $A(t+T) = A(t)$, $B(t+T) = B(t)$. This section only studies uniformly completely controllable (2.4)-(2.5) with the period $T$.

### 2.4.1 Feedback controller design based on the Floquet-Lyapunov theory

Let $\Phi(t, \tau)$ be the state transition matrix of (2.4)-(2.5). The matrix $\Phi(T, 0)$ is called **monodromy** matrix. Furthermore, let

$$W_r(t_1, t_0) := \int_{t_0}^{t_1} \Phi(t_1, t) B(t) B^T(t) \Phi^T(t_1, t) dt.$$

The following proposition shows the relation between controllability of system (2.4)-(2.5) and the monodromy matrix [41].

**Proposition 2.14** *Linear system* (2.4)-(2.5) *is completely controllable if and only if linear discrete time system*

$$x_\epsilon((i+1)T) = \Phi(T, 0) x_\epsilon(iT) + W_r(T, 0) u_\epsilon(iT) \tag{2.48}$$

*is controllable.*

Let us consider the monodromy eigenvalue assignment by using the concept of **sampled state periodic hold** control [41] of the form

$$u_\epsilon(t) = F_s(t)x_\epsilon(iT),\ t \in [iT, (i+1)T),\ i \in \boldsymbol{Z}, \tag{2.49}$$
$$F_s(t+T) = F_s(t). \tag{2.50}$$

Applying a feedback (2.49)-(2.50) into system (2.4)-(2.5), the state transition satisfies

$$x_\epsilon(iT+\sigma) = \left[\Phi(\sigma,0) + \int_0^\sigma \Phi(\sigma,\tau)B(\tau)F_s(\tau)d\tau\right] x_\epsilon(iT),\ \sigma \in [0,T),\ i \in \boldsymbol{Z}. \tag{2.51}$$

If we set

$$F_s(t) = B^T(t)\Phi^T(T,t)F,$$

(2.51) yields a linear time invariant discrete system

$$x_\epsilon((i+1)T) = A_d x_\epsilon(iT), \tag{2.52}$$

where $A_d := \Phi(T,0) + W_r(T,0)F$. By proposition 2.14, linearized system (2.4)-(2.5) is completely controllable if and only if linear discrete time system (2.48) is controllable. Since linear discrete system (2.48) is controllable, eigenvalues of the matrix $A_d$ of closed-loop (2.52) of system (2.48) can be assigned arbitrary values [32]. Hence if we assign all eigenvalues of $A_d$ in the unit circle,

$$x_\epsilon(iT) \to 0 \quad (i \to \infty). \tag{2.53}$$

We note that if linearized system (2.4)-(2.5) is controllable on $[0,T]$, and if

$$F_s(t) = B^T(t)\Phi^T(T,t)W_r^{-1}(T,0)(A_d - \Phi(T,0)), \tag{2.54}$$

we have

$$||x_\epsilon(iT+\sigma)|| \to 0 \quad (i \to \infty),\ \sigma \in [0,T) \tag{2.55}$$

This means that the origin of the resulting closed loop of system (2.4)-(2.5)

$$\dot{x}_\epsilon(t) = A(t)x_\epsilon(t) + B(t)F_s(t)x_\epsilon(iT), \quad t \in [iT, (i+1)T),\ i \in \boldsymbol{Z}$$

can be made asymptotically stabilizable.

From now on, let us prove the above fact. Since $(x^*(t), u^*(t))$ is a controllable trajectory, $P^c_{(x^*(t),u^*(t))}$ is bounded. Hence $A(t)$ and $B(t)$ are bounded on $\boldsymbol{R}$. Thus there exist $k_A > 0$ and $k_B > 0$ such that

$$||A(t)|| \le k_A, \quad ||B(t)|| \le k_B.$$

Furthermore by the Peano-Baker formula [102], $\Phi(\sigma,\tau)$ can be expressed by

$$\begin{aligned}\Phi(\sigma,\tau) = I + \int_\tau^\sigma A(s_1)ds_1 + \int_\tau^\sigma \int_\tau^{s_1} A(s_1)A(s_2)ds_2ds_1 + \cdots \\ + \int_\tau^\sigma \int_\tau^{s_1} \cdots \int_\tau^{s_{l-1}} A(s_1)A(s_2)\cdots A(s_l)ds_l \cdots ds_2ds_1 + \cdots .\end{aligned}$$

Therefore

$$||\Phi(\sigma,\tau)|| \leq \exp((\sigma-\tau)k_A).$$

Hence (2.51) implies that

$$||x_\epsilon(iT+\sigma)|| \leq \left(\exp(Tk_A) + k_B T \exp(Tk_A)\cdot \max_{0\leq\tau\leq\sigma} ||F_s(\tau)||\right) ||x_\epsilon(iT)|| \quad (2.56)$$

Furthermore

$$||F_s(\tau)|| \leq k_B \exp(Tk_A)||W_r^{-1}(T,0)|| \left(||A_d|| + \exp(Tk_A)\right). \quad (2.57)$$

Note that $||W_r^{-1}(T,0)||$ is bounded because linearized system (2.4)-(2.5) is controllable on $[0,T]$. Eqs. (2.56)-(2.57) implies that there exists a constant $c > 0$ such that

$$||x_\epsilon(iT+\sigma)|| \leq c||x_\epsilon(iT)|| \quad (2.58)$$

Since (2.53) is satisfied, (2.58) yields (2.55).

**Remark 2.9** *To use* (2.54)*, it is necessary that* $W_r(T,0)$ *is invertible. This means that linearized system* (2.4)-(2.5) *is controllable on* $[0,T]$*. By corollary 2.7, a linearized system of algebraically controllable system* (2.1)-(2.2) *along a periodic controllable trajectory is uniformly completely controllable, that is, completely controllable. Although the concept of complete controllability is not clear how long a time interval is needed for controllability, by observing the proofs of lemma 2.16 and theorem 2.6, we can take arbitrarily small an time interval for controllability. Therefore we can conclude that the linearized system* (2.4)-(2.5) *is controllable on* $[0,T]$*.* ■

To use the sampled state periodic hold control (2.49)-(2.50), where $F_s(t)$ is defined as (2.54), we have to calculate the state transition matrix $\Phi(T,t)$ for all $0 \leq t < T$. The next subsection elaborates another feedback controller design method by using a monodromy matrix.

### 2.4.2 Feedback controller design based on LQ optimal control theory

Let us consider a periodic linear quadratic (LQ) optimal control

$$\begin{aligned} \min_{u_\epsilon} \quad & \frac{1}{2}\int_0^\infty x_\epsilon^T(t)C^T(t)C(t)x_\epsilon(t) + u_\epsilon^T(t)R(t)u_\epsilon(t)dt \\ \text{subject to} \quad & (2.4)-(2.5),\ x_\epsilon(0) = x_{\epsilon,0}, \end{aligned} \tag{2.59}$$

where $R(t) = R^T(t) > 0$ is periodic with the period $T$ and continuous. It is known (see theorem 2 in [3] and see theorem 4 in [4]) that if system (2.4)-(2.5) is completely controllable and completely observable, the optimal control $u_\epsilon^{\mathrm{opt}}(t)$ is **uniquely** given by

$$u_\epsilon^{\mathrm{opt}}(t) = -R^{-1}(t)B^T(t)P(t)x_\epsilon(t), \tag{2.60}$$

where $P(t)$ is the unique positive definite periodic solution of the periodic **Riccati differential equation** (PRDE)

$$-\dot{P}(t) = A^T(t)P(t) + P(t)A^T(t) - P(t)B(t)R^{-1}(t)B^T(t)P(t) + C^T(t)C(t). \tag{2.61}$$

Moreover, since system (2.4)-(2.5) is periodic, and completely controllable and observable, the origin of closed-loop

$$\dot{x}_\epsilon = \left(A(t) - B(t)R^{-1}(t)B^T(t)P(t)\right)x_\epsilon$$

is exponentially stable (see theorem 2 in [3]).

**Interpretation 2.3** *Suppose that a given system* (2.1)-(2.2) *is algebraically controllable and observable, and* $(x^*(t), u^*(t))$ *is a periodic controllable and observable trajectory. Let* $x^*(t)$ *be the reference trajectory of system* (2.1)-(2.2). *Then by corollaries 2.7 and 2.12, as mentioned in section 2.1, if we apply*

$$u(t) = u^*(t) - R^{-1}(t)B^T(t)P(t)(x(t) - x^*(t))$$

*into system* (2.1)-(2.2)*, the actual trajectory* $x(t)$ *locally exponentially approaches the reference trajectory* $x^*(t)$*.*

We note that references [27,31,110] have studied numerical analysis methods to solve (2.61). From now on, we explain a simple method called a **periodic generator method** [27] to solve (2.61). Let $S_1(t), S_2(t) \in \boldsymbol{R}^{n\times n}$. Suppose that $S_1(t)$ is invertible for all $t \in \boldsymbol{R}$ and each element of $S_1(t)$ and $S_2(t)$ is smooth. Then if we put $P(t)$ satisfying (2.61) as $S_2(t)S_1^{-1}(t)$, (2.61) is equivalent to

$$(\dot{S}_2 + A^T(t)S_2 + QS_1)S_1^{-1} - S_2S_1(\dot{S}_1 - A(t)S_1 + B(t)R^{-1}(t)B^T(t)S_2)S_1^{-1} = 0.$$

Hence if

$$\begin{pmatrix} \dot{S}_1 \\ \dot{S}_2 \end{pmatrix} = \underbrace{\begin{pmatrix} A(t) & -B(t)R^{-1}(t)B^T(t) \\ -Q(t) & -A^T(t) \end{pmatrix}}_{H(t)} \begin{pmatrix} S_1 \\ S_2 \end{pmatrix} \tag{2.62}$$

is satisfied, PRDE (2.61) is also satisfied. The matrix $H(t)$ in (2.62) is called the Hamiltonian matrix corresponding to PRDE (2.61). Let $\Phi_H(t,\tau)$ denote the transition matrix of $H(t)$. The periodic generator method is composed of the following procedures.

1. Compute the monodromy matrix $\Phi_H(T,0)$.
2. Compute the ordered real Schur form such that

   $$U^T(0)\Phi_H(T,0)U(0) = \begin{pmatrix} \Theta_{11} & \Theta_{12} \\ 0 & \Theta_{22} \end{pmatrix}, \tag{2.63}$$

   where $\Theta_{11}$ has $n$ eigenvalues inside the unit circle.
3. Set $U(t) := \Phi_H(t,0)U(0)$ and partition $U(t)$ into $n \times n$ blocks

   $$U(t) = \begin{pmatrix} U_{11}(t) & U_{12}(t) \\ U_{21}(t) & U_{22}(t) \end{pmatrix}.$$

   Compute $P(t) = U_{21}(t)U_{11}^{-1}(t)$.

Note that $P(t)$ is the periodic solution of (2.61). In fact, by definition of $U(t)$, the matrix $\begin{pmatrix} U_{11}(t) \\ U_{21}(t) \end{pmatrix}$ satisfies (2.62). Moreover, since $U(T) = \Phi(T,0)U(0)$ is equivalent to

$$U(T) = \begin{pmatrix} U_{11}(0)\Theta_{11} & U_{11}(0)\Theta_{12} + U_{12}(0)\Theta_{22} \\ U_{21}(0)\Theta_{11} & U_{21}(0)\Theta_{12} + U_{22}(0)\Theta_{22} \end{pmatrix},$$

we have

$$\begin{pmatrix} U_{11}(T) \\ U_{21}(T) \end{pmatrix} = \begin{pmatrix} U_{11}(0) \\ U_{21}(0) \end{pmatrix} S_{11}.$$

Hence

$$P(T) = U_{21}(T)U_{11}^{-1}(T) = U_{21}(0)U_{11}^{-1}(0) = P(0).$$

In chapter 4, we apply the above periodic LQ optimal control for a trajectory tracking control.

## 2.5 Coordinate transformation of error system

Let us consider system (2.3) again. Now we perform a coordinate transformation such as

$$\tilde{x}_\epsilon = \alpha(t, x_\epsilon), \tag{2.64}$$

where $\alpha : \boldsymbol{R} \times \boldsymbol{R}^n \to \boldsymbol{R}^n$ is smooth, $\alpha(t, 0) = 0$, $\frac{\partial \alpha}{\partial t}(t, 0) = 0$, and $S(t) := \frac{\partial \alpha}{\partial x_\epsilon}(t, 0)$ is invertible for all $t \geq 0$. Although, rigorously, the dynamics of $\tilde{x}_\epsilon$ obey

$$\dot{\tilde{x}}_\epsilon = \frac{\partial \alpha}{\partial t}(t, x_\epsilon) + \frac{\partial \alpha}{\partial x_\epsilon}(t, x_\epsilon)(f(x_\epsilon + x^*(t), u_\epsilon + u^*(t)) - f(x^*(t), u^*(t))), \tag{2.65}$$

since we can regard the dynamics of $x_\epsilon$ as (2.4) around $x_\epsilon = 0$, we can consider that the dynamics of $\tilde{x}_\epsilon$ around $x_\epsilon = 0$ is approximated by

$$\dot{\tilde{x}}_\epsilon = \frac{\partial \alpha}{\partial t}(t, 0) + \frac{\partial^2 \alpha}{\partial t \partial x_\epsilon} x_\epsilon(t, 0) + \frac{\partial \alpha}{\partial x_\epsilon}(t, 0)(A(t)x_\epsilon + B(t)u_\epsilon).$$

Furthermore, around $x_\epsilon = 0$, since $\alpha(t, 0) = 0$, (2.64) is approximated by

$$\tilde{x}_\epsilon = S(t)x_\epsilon. \tag{2.66}$$

Since $\frac{\partial \alpha}{\partial t}(t, 0) = 0$, and $S(t)$ is invertible, around $x_\epsilon = 0$, the dynamics of $\tilde{x}_\epsilon$ is approximated by

$$\dot{\tilde{x}}_\epsilon = \underbrace{\left(S(t)A(t)S(t)^{-1} + \dot{S}(t)S^{-1}\right)}_{\tilde{A}(t)} \tilde{x}_\epsilon + \underbrace{S(t)B(t)}_{\tilde{B}(t)} u_\epsilon. \tag{2.67}$$

If we can design a feedback gain $K(t)$ such that the origin of the closed-loop

$$\dot{\tilde{x}}_\epsilon = \left(\tilde{A}(t) + \tilde{B}(t)K(t)\right) \tilde{x}_\epsilon, \tag{2.68}$$

is exponentially stable, by applying the feedback control $u_\epsilon = K(t)\tilde{x}_\epsilon$ into system (2.65), the origin of the resulting closed-loop is locally exponentially stable [48]. From (2.66), $x_\epsilon = S(t)^{-1}\tilde{x}_\epsilon$ around $x_\epsilon = 0$. Hence if $\tilde{x}_\epsilon = 0$ of (2.66) is exponentially stable, $x_\epsilon = 0$ of (2.4) is also exponentially stable. Correspondingly, using the same $K(t)$, if we apply

$$u = u^*(t) + K(t)\tilde{x}_\epsilon$$

into system (2.1)-(2.2), the actual trajectory $x(t)$ locally exponentially approaches the reference trajectory $x^*(t)$. As mentioned in section 2.1, complete controllability of system (2.67) is strongly related with exponential stabilizability of the origin of system (2.67). For this reason, the the following lemma [50] is valuable.

**Lemma 2.20** *System* (2.4) *is completely controllable if and only if system* (2.67) *is completely controllable.*

**Proof** Let

$$W(t_1,t) := \int_t^{t_1} \Phi(t,\tau)B(\tau)B^T(\tau)\Phi^T(t,\tau)d\tau.$$

Since

$$W(t_1,t) = \Phi(t,t_0)\int_t^{t_1} \Phi(t_0,\tau)B(\tau)B^T(\tau)\Phi^T(t_0,\tau)d\tau\Phi^T(t,t_0),$$

differentiating $W(t_1,t)$ with respect to $t$, we have

$$\begin{aligned}\frac{dW}{dt}(t_1,t) = -\Phi(t,t)B(t)B^T(t)\Phi^T(t,t) + \int_t^{t_1} A(t)\Phi(t,\tau)B(\tau)B^T(\tau)\Phi^T(t,\tau)d\tau \\ + \int_t^{t_1} \Phi(t,\tau)B(\tau)B^T(\tau)\Phi^T(t,\tau)A^T(\tau)d\tau\end{aligned}$$

Since $\Phi(t,t) = I$, the matrix $W(t_1,t)$ satisfies

$$\frac{dW}{dt} = A(t)W + WA^T(t) - B(t)B(t)^T, \tag{2.69}$$

$$W(t_1,t_1) = 0. \tag{2.70}$$

Moreover, if we constitute $\tilde{W}(t_1,t)$ corresponding to system (2.67), $\tilde{W}(t_1,t)$ satisfies

$$\frac{d\tilde{W}}{dt} = \tilde{A}(t)\tilde{W} + \tilde{W}\tilde{A}^T(t) - \tilde{B}(t)\tilde{B}(t)^T, \tag{2.71}$$

$$\tilde{W}(t_1,t_1) = 0. \tag{2.72}$$

On the other hand, if we define

$$\bar{W}(t) := S(t)W(t_1,t)S^T(t),$$

by (2.69), $\bar{W}(t)$ satisfies

$$\frac{d\bar{W}}{dt} = \tilde{A}(t)\bar{W} + \bar{W}\tilde{A}^T(t) - \tilde{B}(t)\tilde{B}^T(t). \tag{2.73}$$

Furthermore, (2.70) yields

$$\bar{W}(t_1) = 0. \tag{2.74}$$

By (2.71), (2.72), and (2.73), (2.74), $\tilde{W}$ and $\bar{W}$ obey the same linear ordinary differential equation and initial condition. Therefore by the theorem on uniqueness of solution of ordinary differential equation,

$$\bar{W}(t) = \tilde{W}(t_1, t).$$

Hence

$$\tilde{W}(t_1, t_0) = S(t)W(t_1, t_0)S^T(t).$$

Since $S(t)$ is invertible, $W(t_1, t_0)$ is invertible if and only if $\tilde{W}(t_1, t_0)$ is invertible. □

By theorem 2.6, corollary 2.7, and lemma 2.20, we have the following corollary.

**Corollary 2.15** *Suppose that system* (2.1)-(2.2) *is algebraically controllable. Let* $(x^*(t), u^*(t))$ *be any (periodic) controllable trajectory. Then linear system* (2.67) *is (uniformly) completely controllable.*

**Example 2.9** Let us go back to example 2.2. Let

$$\begin{pmatrix} x_\epsilon \\ y_\epsilon \\ \theta_\epsilon \end{pmatrix} = \begin{pmatrix} x - x^* \\ y - y^* \\ \theta - \theta^* \end{pmatrix}, \quad \begin{pmatrix} u_{1,\epsilon} \\ u_{2,\epsilon} \end{pmatrix} = \begin{pmatrix} u_1 - u_1^* \\ u_2 - u_2^* \end{pmatrix}.$$

Then Eq. (2.29) yields

$$\begin{pmatrix} \dot{x}_\epsilon \\ \dot{y}_\epsilon \\ \dot{\theta}_\epsilon \end{pmatrix} = \begin{pmatrix} (\cos(\theta^*(t) + \theta_\epsilon) - \cos\theta^*(t))u_1^*(t) + \cos(\theta^*(t) + \theta_\epsilon)u_{1,\epsilon} \\ (\sin(\theta^*(t) + \theta_\epsilon) - \sin\theta^*(t))u_1^*(t) + \sin(\theta^*(t) + \theta_\epsilon)u_{1,\epsilon} \\ u_{2,\epsilon} \end{pmatrix}. \tag{2.75}$$

Linearizing at $(x_\epsilon, y_\epsilon, \theta_\epsilon) = (0, 0, 0)$ and $(u_{1,\epsilon}, u_{2,\epsilon}) = (0, 0)$, we have

$$\begin{pmatrix} \dot{x}_\epsilon \\ \dot{y}_\epsilon \\ \dot{\theta}_\epsilon \end{pmatrix} = \begin{pmatrix} 0 & 0 & -\sin(\theta^*(t))u_1^*(t) \\ 0 & 0 & \cos(\theta^*(t))u_1^*(t) \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_\epsilon \\ y_\epsilon \\ \theta_\epsilon \end{pmatrix} + \begin{pmatrix} \cos(\theta^*(t)) & 0 \\ \sin(\theta^*(t)) & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u_{1,\epsilon} \\ u_{2,\epsilon} \end{pmatrix}. \tag{2.76}$$

Now we apply a coordinate transformation

$$\begin{aligned} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} &= \begin{pmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_\epsilon \\ y_\epsilon \\ \theta_\epsilon \end{pmatrix} \\ &= \begin{pmatrix} \cos(\theta^*(t) + \theta_\epsilon) & \sin(\theta^*(t) + \theta_\epsilon) & 0 \\ -\sin(\theta^*(t) + \theta_\epsilon) & \cos(\theta^*(t) + \theta_\epsilon) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_\epsilon \\ y_\epsilon \\ \theta_\epsilon \end{pmatrix} \end{aligned} \tag{2.77}$$

into (2.75). At $(x_\epsilon, y_\epsilon, \theta_\epsilon) = (0,0,0)$, the above transformation can be approximated as

$$\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} \cos(\theta^*(t)) & \sin(\theta^*(t)) & 0 \\ -\sin(\theta^*(t)) & \cos(\theta^*(t)) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_\epsilon \\ y_\epsilon \\ \theta_\epsilon \end{pmatrix}.$$

Hence then around $(x_\epsilon, y_\epsilon, \theta_\epsilon) = (0,0,0)$ and $(u_{1,\epsilon}, u_{2,\epsilon}) = (0,0)$, linearized system (2.76) is transformed into

$$\begin{pmatrix} \dot{e}_1 \\ \dot{e}_2 \\ \dot{e}_3 \end{pmatrix} = \begin{pmatrix} 0 & u_2^*(t) & 0 \\ -u_2^*(t) & 0 & u_1^*(t) \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u_{1,\epsilon} \\ u_{2,\epsilon} \end{pmatrix} \tag{2.78}$$

By example 2.2, system (2.29) is algebraically controllable. Hence corollary 2.15 implies that if $(x^*(t), y^*(t), \theta^*(t), u_1^*(t), u_2^*(t))$ is a (periodic) controllable trajectory, linear system (2.78) is (uniformly) completely controllable.

Fig. 2.1 illustrates the coordinate transformation (2.77). We note that the transformation (2.77) has been frequently used for trajectory tracking control of nonholonomic mobile robot (2.29) [5, 39, 44, 45, 53].

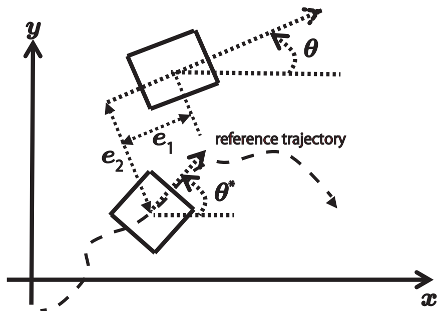



Figure 2.1: Coordinate transformation (2.77)

■

Similarly, we have the following corollary.

**Corollary 2.16** *Suppose that system* (2.1)-(2.2) *is algebraically observable. Let* $(x^*(t), u^*(t))$ *be any (periodic) observable trajectory. Then linear system* (2.67) *with output* $y_\epsilon = C(t)S^{-1}(t)\tilde{x}_\epsilon$ *is (uniformly) completely observable.*

## 2.6 The relation of algebraic controllability and accessibility

This section shows that nonlinear system (2.1)-(2.2) is algebraically controllable if and only if the system is accessible. Accessibility is defined by using a concept of autonomous variable [16]. Let $\mathcal{X}$ denote the subspace of $\mathcal{E}_{(x,u)}$ defined as

$$\mathcal{X} := \operatorname{span}_{\mathcal{M}_{(x,u)}} \{dx_i, i = 1, \cdots, n\}.$$

**Definition 2.21** *A one-form $\omega \in \mathcal{X}$ is called an* **autonomous variable** *of system* (2.1)-(2.2) *if there exists $\alpha \in \mathcal{D}_{(x,u)}$, $\deg \alpha \geq 1$ such that*

$$\alpha\omega = 0. \tag{2.79}$$

**Definition 2.22** *System* (2.1)-(2.2) *is called* **accessible** *if there does not exist any non-zero autonomous variable in $\mathcal{X}$.*

In general, if a given system (2.1)-(2.2) is not accessible, there exist several autonomous variables.

**Example 2.10** Let us consider

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = -x_1 + u, \\ \dot{x}_3 = x_4, \\ \dot{x}_4 = -x_3 + u, \end{cases} \tag{2.80}$$

where $x_1, \cdots, x_4$ are state variables and $u$ is an input variable. Putting $\omega := d(x_3 - x_1)$, (2.80) yields

$$(1 + \frac{d^2}{dt^2})\omega = 0.$$

Hence $\omega$ is an autonomous variable of system (2.80).

On the other hand, putting $\tilde{\omega} := d(x_2 - x_4)$, (2.80) yields

$$(1 + \frac{d^2}{dt^2})\tilde{\omega} = 0.$$

Therefore $\tilde{\omega}$ is also an autonomous variable of system (2.80).

Note that

$$\begin{aligned} (1 + \frac{d^2}{dt^2})\omega &= -\frac{d}{dt}\omega_1 - \omega_2 + \frac{d}{dt}\omega_3 + \omega_4, \\ (1 + \frac{d^2}{dt^2})\tilde{\omega} &= -\omega_1 + \frac{d}{dt}\omega_2 + \omega_3 - \frac{d}{dt}\omega_4, \end{aligned}$$

where

$$\begin{aligned} \omega_1 &:= d\dot{x}_1 - dx_2, \\ \omega_2 &:= d\dot{x}_2 + dx_1 - du, \\ \omega_3 &:= d\dot{x}_3 - dx_4, \\ \omega_4 &:= d\dot{x}_4 + dx_3 - du. \end{aligned}$$

■

The concept of relative degree can be extended to one-form [16] (see definition F.1 in appendix F).

**Definition 2.23** *The relative degree $r$ of a one-form $\omega \in \mathcal{E}_{(x,u)}$ is defined as*

$$r := \inf\{k \in \boldsymbol{Z} \mid \operatorname{span}_{\mathcal{M}_{(x,u)}}\{\omega, \cdots, \omega^{(k)}\} \not\subset \mathcal{X}\},$$

*In particular, we say that $\phi$ has* **finite relative degree** *if $r$ is finite and that $\phi$ has* **infinite relative degree** *if $r = \infty$.*

**Remark 2.10** *Let $\phi \in \mathcal{M}_{(x,u)}$. If $d\phi$ has infinite relative degree, $\phi^{(k)}$, $k \geq 0$ is not influenced by a control input $u$.* ■

The following propositions can be found in [16].

**Proposition 2.17** *A one-form $\omega \in \mathcal{X}$ is an autonomous variable if and only if it has an infinite relative degree.*

**Proof** Suppose that $\omega \in \mathcal{X}$ has an infinite relative degree. Since $\dim \mathcal{X} = n$,

$$\dim \operatorname{span}_{\mathcal{M}_{(x,u)}}\{\omega, \dot{\omega}, \cdots\} \leq n.$$

Hence there exists $\alpha \in \mathcal{D}_{(x,u)}$, $\deg \alpha \geq 1$ satisfying (2.79).

Conversely, if $\omega$ has finite relative degree,

$$\dim \operatorname{span}_{\mathcal{M}_{(x,u)}}\{\omega, \cdots, \omega^{(k-1)}\} = k$$

for any $k \geq 1$. This means that there does not exist $\alpha \in \mathcal{D}_{(x,u)}$, $\deg \alpha \geq 1$ satisfying (2.79). □

From now on, we relate algebraic controllability and accessibility of system (2.1)-(2.2). On this account, we give some mathematical facts.

**Lemma 2.24** *Let $A \in \mathcal{D}_{(x,u)}^{m\times m}$. If there exists a matrix $B \in \mathcal{D}_{(x,u)}^{m\times m}$ such that*

$$AB = I_m, \tag{2.81}$$

*then $A$ is unimodular.*

**Proof** By definition of rank (see section 2.2), we have

$$m = \operatorname{rank} I_m = \operatorname{rank}(AB) \leq \operatorname{rank} A \leq m.$$

Hence $\operatorname{rank} A = m$. Eq. (2.81) implies $ABA = A$, so that $A(BA - I_m) = 0$. Since $\operatorname{rank} A = m$,

$$BA = I_m.$$

Therefore $A$ is unimodular. □

As a special case of proposition 2.1, the following corollary can be obtained [56, 114].

**Proposition 2.18** *Let $R \in \mathcal{D}^{m\times n}_{(x,u)}$ be a matrix with full row rank. Then there exists a unimodular matrix $U \in \mathcal{D}^{n\times n}_{(x,u)}$ such that*

$$R = \begin{pmatrix} G & 0 \end{pmatrix} U, \tag{2.82}$$

*where*

$$G := \begin{pmatrix} g_{11} & & & \\ g_{21} & g_{22} & & \\ \vdots & \vdots & \ddots & \\ g_{m1} & g_{m2} & \cdots & g_{mm} \end{pmatrix} \in \mathcal{D}^{m\times m}_{(x,u)}. \tag{2.83}$$

*Moreover, if $\deg g_{kk} > 0$, the polynomials $g_{ki}$, $i = 1, \cdots, k-1$ are of lower degree than the polynomials $g_{kk}$ for $k = 1, \cdots, m$. If $\deg g_{kk} = 0$, the polynomials $g_{ki} = 0$, $i = 1, \cdots, k-1$.*

**Definition 2.25** *Let $R \in \mathcal{D}^{m\times n}_{(x,u)}$. A matrix $A \in \mathcal{D}^{m\times m}_{(x,u)}$ is called a* **left divisor** *of $R$ if there exists a matrix $B \in \mathcal{D}^{m\times n}_{(x,u)}$ such that*

$$R = AB.$$

**Definition 2.26** *Let $R \in \mathcal{D}^{m\times n}_{(x,u)}$. A matrix $A \in \mathcal{D}^{m\times m}_{(x,u)}$ is called a* **greatest left divisor** *(gld) of $R$ if the following conditions are satisfied:*

1. *$A$ is a left divisor of $R$.*
2. *If $A' \in \mathcal{D}^{m\times m}_{(x,u)}$ is also a left divisor of $R$, then there exists a matrix $B \in \mathcal{D}^{m\times m}_{(x,u)}$ such that*

$$A = A'B.$$

We can obtain the following proposition [56].

**Proposition 2.19** *Let $R \in \mathcal{D}^{m\times n}_{(x,u)}$ be a matrix with full row rank. The matrix $G \in \mathcal{D}^{m\times m}_{(x,u)}$ satisfying* (2.82) *is a greatest left divisor of $R$.*

**Definition 2.27** *Let $R \in \mathcal{D}^{m\times n}_{(x,u)}$ be a matrix with full row rank. The matrix $R$ is called* **left prime** *if there exist matrices $L \in \mathcal{D}^{m\times m}_{(x,u)}$ and $R' \in \mathcal{D}^{m\times n}_{(x,u)}$ such that $R = LR'$, where $L$ is unimodular.*

**Lemma 2.28** *Let $R \in \mathcal{D}^{m\times n}_{(x,u)}$ be a matrix with full row rank. Then the matrix $R$ is left prime if and only if $R$ is hyper-regular.*

**Proof** Suppose that $R$ is left prime. By proposition 2.1, there exist unimodular matrices $U \in \mathcal{D}_{(x,u)}^{m\times m}$ and $V \in \mathcal{D}_{(x,u)}^{n\times n}$ such that

$$URV = \begin{pmatrix}\Delta & 0\end{pmatrix},$$

where $\Delta := \text{diag}\,(1,\cdots,1,\alpha) \in \mathcal{D}_{(x,u)}^{m\times m}$, $0 \neq \alpha \in \mathcal{D}_{(x,u)}$. Then if we denote $V^{-1}$ by $\begin{pmatrix}\tilde{V}_1 & \tilde{V}_2\\ \tilde{V}_3 & \tilde{V}_4\end{pmatrix}$,

$$R = \left(U^{-1}\Delta\right)\begin{pmatrix}\tilde{V}_1 & \tilde{V}_2\end{pmatrix}.$$

Since $R$ is left prime, $U^{-1}\Delta$ is unimodular. Then since $U^{-1}$ is unimodular, $\Delta$ is unimodular. If $\deg\ \alpha \geq 1$, $\Delta$ is not unimodular, so that $\deg\alpha = 0$.

Conversely, suppose that $R$ is hyper-regular, that is, there exist unimodular matrices $U \in \mathcal{D}_{(x,u)}^{m\times m}$ and $V \in \mathcal{D}_{(x,u)}^{n\times n}$ satisfying

$$URV = \begin{pmatrix}I_m & 0\end{pmatrix}.$$

Then by a straightforward calculation, $\begin{pmatrix}R\\ T\end{pmatrix} = \begin{pmatrix}U^{-1} & 0\\ 0 & I_{n-m}\end{pmatrix}V^{-1}$, where $T := \begin{pmatrix}0 & I_{n-m}\end{pmatrix}V^{-1}$. Hence $\begin{pmatrix}R\\ T\end{pmatrix}$ is unimodular, so that there exists $\begin{pmatrix}\tilde{R} & \tilde{T}\end{pmatrix} \in \mathcal{D}_{(x,u)}^{n\times n}$ such that $\begin{pmatrix}R\\ T\end{pmatrix}\begin{pmatrix}\tilde{R} & \tilde{T}\end{pmatrix} = \begin{pmatrix}I_m & \\ & I_{n-m}\end{pmatrix}$. Therefore there exists $\tilde{R} \in \mathcal{D}_{(x,u)}^{n\times m}$ such that $R\tilde{R} = I_m$. Thus if there exist matrices $L \in \mathcal{D}_{(x,u)}^{m\times m}$ and $R' \in \mathcal{D}_{(x,u)}^{m\times n}$ such that $R = LR'$, using the matrix $\tilde{R}$, we have

$$L(R'\tilde{R}) = I_m.$$

Then by lemma 2.24, $L$ is unimodular. □

**Lemma 2.29** *Let $R \in \mathcal{D}_{(x,u)}^{m\times n}$ be a matrix with full row rank. Suppose that $G \in \mathcal{D}_{(x,u)}^{m\times m}$ and $G' \in \mathcal{D}_{(x,u)}^{m\times m}$ are two arbitrary greatest left divisors of $R$. Then there exists a unimodular matrix $T \in \mathcal{D}_{(x,u)}^{m\times m}$ such that*

$$G = G'T.$$

**Proof** By the definition of gld, there exist matrices $T \in \mathcal{D}_{(x,u)}^{m\times m}$ and $T' \in \mathcal{D}_{(x,u)}^{m\times m}$ such that

$$\begin{cases} G = G'T, \\ G' = GT'. \end{cases}$$

This implies that

$$\begin{cases} G(T'T - I_m) = 0, \\ G'(TT' - I_m) = 0. \end{cases} \tag{2.84}$$

On the other hand, since $G$ and $G'$ are left divisors of $R$, there exist $\tilde{R}_1 \in \mathcal{D}_{(x,u)}^{m\times n}$ and $\tilde{R}_2 \in \mathcal{D}_{(x,u)}^{m\times n}$ such that

$$\begin{cases} R = G\tilde{R}_1, \\ R = G'\tilde{R}_2. \end{cases}$$

Since $R$ has full row rank, this implies that

$$m = \text{rank } R \le \text{rank } G, \ \text{rank } G' \le m,$$

that is, rank $G =$ rank $G' = m$. Therefore Eq. (2.84) leads to

$$T'T = TT' = I_m.$$

Thus $T$ is unimodular. □

**Lemma 2.30** *Let $R \in \mathcal{D}_{(x,u)}^{m\times n}$ be a matrix with full row rank. If $R$ is not left prime, then any greatest left divisor of $R$ is not unimodular.*

**Proof** Suppose that $R$ is not left prime. Then by proposition 2.1 and lemma 2.28, there exist unimodular matrices $U \in \mathcal{D}_{(x,u)}^{m\times m}$ and $V \in \mathcal{D}_{(x,u)}^{n\times n}$ such that

$$URV = \begin{pmatrix} \Delta & 0 \end{pmatrix},$$

where $\Delta := \text{diag }(1, \cdots, 1, f)$, $\deg f \ge 1$, so that $\Delta$ is not unimodular. Then if we denote $V^{-1}$ by $\begin{pmatrix} \tilde{V}_1 & \tilde{V}_2 \\ \tilde{V}_3 & \tilde{V}_4 \end{pmatrix}$, we have

$$R = \left(U^{-1}\Delta\right)\begin{pmatrix} \tilde{V}_1 & \tilde{V}_2 \end{pmatrix}.$$

Thus $U^{-1}\Delta$ is a left divisor of $R$. From now on, we show that $U^{-1}\Delta$ is a gld. If we denote $V$ by $\begin{pmatrix} V_1 & V_2 \end{pmatrix}$, we have

$$R\begin{pmatrix} V_1 & V_2 \end{pmatrix} = \begin{pmatrix} U^{-1}\Delta & 0 \end{pmatrix}.$$

Hence $U^{-1}\Delta = RV_1$. If $L \in \mathcal{D}_{(x,u)}^{m\times m}$ is an arbitrary left divisor of $R$, there exists $\tilde{R} \in \mathcal{D}_{(x,u)}^{m\times n}$ such that $R = L\tilde{R}$. Therefore

$$U^{-1}\Delta = L(\tilde{R}V_1).$$

Thus $U^{-1}\Delta$ is a gld of $R$.

Let $G_L \in \mathcal{D}_{(x,u)}^{m\times m}$ be an arbitrary gld of $R$. Then by lemma 2.29, there exists unimodular matrix $T \in \mathcal{D}_{(x,u)}^{m\times m}$ such that

$$G_L = (U^{-1}\Delta)T. \tag{2.85}$$

If $G_L$ is unimodular, Eq. (2.85) implies that $\Delta = UG_LT^{-1}$, so that $\Delta$ is also unimodular. This is a contradiction. □

Now, we are in a position to describe the relation between left primeness of the matrix $P^c_{(x,u)}$ and the absence of autonomous variables of system (2.1)-(2.2). Similar discussion can be found in [56].

**Theorem 2.20** *System* (2.1)-(2.2) *is accessible if and only if* $P^c_{(x,u)}$ *defined by* (2.15) *is left prime.*

**Proof** Suppose that $P^c_{(x,u)}$ is not left prime. Then by propositions 2.18 and 2.19, there exists a gld $G_{(x,u)} \in \mathcal{D}_{(x,u)}^{n\times n}$ of $P^c_{(x,u)}$, where $G_{(x,u)}$ is described as the form (2.83). Thus there exists a matrix $\tilde{P}_{(x,u)} \in \mathcal{D}_{(x,u)}^{n\times(n+m)}$ such that

$$P^c_{(x,u)}\begin{pmatrix} dx \\ du \end{pmatrix} = G_{(x,u)}\tilde{P}_{(x,u)}\begin{pmatrix} dx \\ du \end{pmatrix} = 0. \tag{2.86}$$

Let $\tilde{\omega}_i$ be $i$-th row of $\tilde{P}_{(x,u)}\begin{pmatrix} dx \\ du \end{pmatrix}$. Then Eq. (2.86) is equivalent to

$$\begin{pmatrix} g_{11} & & & \\ g_{21} & g_{22} & & \\ \vdots & \vdots & \ddots & \\ g_{n1} & g_{n2} & \cdots & g_{nn} \end{pmatrix}\begin{pmatrix} \tilde{\omega}_1 \\ \tilde{\omega}_2 \\ \vdots \\ \tilde{\omega}_n \end{pmatrix} = 0. \tag{2.87}$$

Since $P^c_{(x,u)}$ is not left prime and $G_{(x,u)}$ is a gld of $P^c_{(x,u)}$, by lemma 2.30, $G_{(x,u)}$ is not unimodular. Assume that $\deg g_{kk} = 0$, $k = 1, \cdots, n$. Then $g_{ki} = 0$, $i = 1, \cdots, k-1$. Thus $G_{(x,u)}$ is unimodular. This is a contradiction, so that there exists $k$ such that $\deg g_{kk} > 0$ and $\deg g_{ii} = 0$, $i = 1, \cdots, k-1$. Then Eq. (2.87) implies that

$$\begin{aligned} \tilde{\omega}_i &= 0, \quad i = 1, \cdots, k-1, \\ g_{kk}\tilde{\omega}_k &= 0. \end{aligned}$$

Since (2.86) and (2.87) are equivalent and $\deg g_{kk} > 0$, we can conclude $\tilde{\omega}_k \in \mathcal{X}$. Therefore the nonzero differential one-form $\tilde{\omega}_k$ is an autonomous variable of system (2.1)-(2.2).

Conversely, suppose that there exists an autonomous variable $\omega \in \mathcal{X}$ of system (2.1)-(2.2). Then by definition 2.21, there exists $\alpha \in \mathcal{D}_{(x,u)}$, $\deg\alpha \geq 1$ satisfying (2.79). Then there exist $\beta_1, \cdots, \beta_n \in \mathcal{D}_{(x,u)}$ such that

$$\alpha\omega = \beta_1\omega_1 + \cdots + \beta_n\omega_n, \tag{2.88}$$

where $\omega_i$ be $i$-th row of $P^c_{(x,u)}\begin{pmatrix} dx \\ du \end{pmatrix}$. In fact, $P^c_{(x,u)}\begin{pmatrix} dx \\ du \end{pmatrix} = 0 \Leftrightarrow \omega_1 = 0, \cdots, \omega_n = 0$ are constraints on one-forms from $\dot{x} = f(x,u)$. Other constraints on one-forms from $\dot{x} = f(x,u)$ are expressed by $\beta_1\omega_1 + \cdots + \beta_n\omega_n = 0$ for some $\beta_1, \cdots, \beta_n \in \mathcal{D}_{(x,u)}$. If there do not exist $\beta_1, \cdots, \beta_n \in \mathcal{D}_{(x,u)}$ satisfying (2.88), it contradicts with the fact. Without loss of generality, we assume that $\deg\beta_1 = \min_{1\leq i\leq n}\{\deg\beta_i,\ \beta_i \neq 0\}$.

**Case 1:** Suppose that $\deg\beta_1 = 0$. Then

$$\begin{pmatrix} \alpha\omega \\ \omega_2 \\ \vdots \\ \omega_n \end{pmatrix} = \begin{pmatrix} \beta_1 & \beta_2 & \cdots & \beta_n \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \vdots \\ \omega_n \end{pmatrix} \Leftrightarrow \begin{pmatrix} \omega_1 \\ \omega_2 \\ \vdots \\ \omega_n \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{\alpha}{\beta_1} & -\frac{\beta_2}{\beta_1} & \cdots & -\frac{\beta_n}{\beta_1} \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}}_{L_{(x,u)}} \begin{pmatrix} \omega \\ \omega_2 \\ \vdots \\ \omega_n \end{pmatrix}$$

Since $\omega$ can be expressed as $Q^1_{(x,u)}\begin{pmatrix} dx \\ du \end{pmatrix}$, where $Q^1_{(x,u)}$ is a some matrix contained in $\mathcal{D}^{1\times(n+m)}_{(x,u)}$, we have

$$P^c_{(x,u)} = L_{(x,u)}\begin{pmatrix} Q^1_{(x,u)} \\ P^2_{(x,u)} \\ \vdots \\ P^n_{(x,u)} \end{pmatrix},$$

where $P^i_{(x,u)}$ denotes $i$-th row of $P^c_{(x,u)}$. Since $L_{(x,u)} \in \mathcal{D}^{n\times n}_{(x,u)}$ is not unimodular, $P^c_{(x,u)}$ is not left prime.

**Case 2:** Suppose that $\deg\beta_1 > 0$. Since $\mathcal{D}_{(x,u)}$ is an Euclidean domain, there exist $\tilde{\beta}_2, \cdots, \tilde{\beta}_n$, $r_2, \cdots, r_n \in \mathcal{D}_{(x,u)}$, where $\deg r_2, \cdots, \deg r_n < \deg\beta_1$ such that

$$\alpha\omega = \beta_1(\omega_1 + \tilde{\beta}_2\omega_2 + \cdots + \tilde{\beta}_n\omega_n) + r_2\omega_2 + \cdots + r_n\omega_n.$$

Since $\omega$ is contained in $\mathcal{X}$ and $\omega_i$ is $i$-th row of $P^c_{(x,u)}\begin{pmatrix} dx \\ du \end{pmatrix}$, $\deg\alpha > \deg\beta_1$. Thus there exist $\tilde{\alpha}$, $r \in \mathcal{D}_{(x,u)}$ such that $\alpha = \beta_1\tilde{\alpha} + r$, $\deg\tilde{\alpha} \geq 1$, and $\deg r < \deg\beta_1$. Therefore, since $\mathcal{D}_{(x,u)}$ is a domain, we obtain

$$\tilde{\alpha}\omega = \omega_1 + \tilde{\beta}_2\omega_2 + \cdots + \tilde{\beta}_n\omega_n.$$

Hence similarly to the case 1, we can show that $P^c_{(x,u)}$ is not left prime. □

By lemma 2.28 and theorem 2.20, we can relate algebraic controllability and accessibility.

**Corollary 2.21** *System* (2.1)-(2.2) *is algebraically controllable if and only if the system is accessible.*

Therefore by theorem 2.6, corollary 2.7, and corollary 2.21, we have the following corollary.

**Corollary 2.22** *Suppose that system* (2.1)-(2.2) *is accessible. Then every linearized system* (2.4)-(2.5) *along any (periodic) controllable trajectory* $(x^*(t), u^*(t))$ *of system* (2.1)-(2.2) *is (uniformly) completely controllable.*

## 2.7 Algebraic controllability of mechanical control systems

Algebraic controllability of a given nonlinear system (2.1)-(2.2) can be examined by elementary matrix operations for a polynomial matrix derived from the given system. However many calculations might be required for checking whether or not a given nonlinear system is algebraically controllable. This is troublesome for practical applications.

In order to resolve the problem, the section restricts attention to a mechanical control system and gives a low computational complexity for checking whether or not the system is algebraically controllable.

Let us consider the mechanical control system

$$M(q)\ddot{q} = \underbrace{C(q,\dot{q}) + B(q)u}_{g(q,\dot{q},u)}, \tag{2.89}$$

where $q \in \boldsymbol{R}^n$ and $u \in \boldsymbol{R}^m$ denote configuration and input variables, respectively, $M(q) \in \boldsymbol{R}^{n\times n}$ is invertible at all $q \in \boldsymbol{R}^n$. Here, each entry of $M(q)$ and $g(q,\dot{q},u)$ are meromorphic with respect to each variable. Although we can transform (2.89) into the form (2.1), if each element of $M(q)$ is a complicated function with respect to $q$, the calculation of the inverse matrix of $M(q)$ may be very hard. Hence we define algebraic controllability of mechanical control systems (2.89) without using the relation $\ddot{q} = M(q)^{-1}g(q,\dot{q},u)$. To this end, we need some mathematical preliminaries.

Let $\tilde{\mathcal{M}}_{(q,u)}$ denote the field of all meromorphic functions depending on a finite number of variables of $\{q_i^{(l)}, u_j^{(l)} \,|\, 1 \le i \le n,\ 1 \le j \le m,\ l \ge 0\}$. Here we do not use the relation $\ddot{q} = M^{-1}(q)g(q, \dot{q}, u)$ for the above mentioned reason. For any $\phi(q, \dot{q}, \cdots, u, \dot{u}, \cdots) \in \tilde{\mathcal{M}}_{(q,u)}$ we define

$$\dot{\phi}(q, \dot{q}, \cdots, u, \dot{u}, \cdots) := \sum_{l \geq 0} \left( \frac{\partial \phi}{\partial q^{(l)}} q^{(l+1)} + \frac{\partial \phi}{\partial u^{(l)}} u^{(l+1)} \right).$$

A vector space $\tilde{\mathcal{E}}_{(q,u)}$ of differential one forms spanned over $\tilde{\mathcal{M}}_{(q,u)}$ is defined as

$$\tilde{\mathcal{E}}_{(q,u)} := \operatorname{span}_{\tilde{\mathcal{M}}_{(q,u)}} \left\{ dq_i^{(l)}, du_j^{(l)} \,\middle|\, 1 \leq i \leq n, 1 \leq j \leq m, l \geq 0 \right\}.$$

Then for any $\phi \in \tilde{\mathcal{M}}_{(q,u)}$, differential $d : \tilde{\mathcal{M}}_{(q,u)} \to \tilde{\mathcal{E}}_{(q,u)}$ is defined as

$$d\phi := \sum_{l \geq 0} \left( \frac{\partial \phi}{\partial q^{(l)}} dq^{(l)} + \frac{\partial \phi}{\partial u^{(l)}} du^{(l)} \right).$$

Let $\tilde{\mathcal{D}}_{(q,u)} := \tilde{\mathcal{M}}_{(q,u)} \left[ \frac{d}{dt} \right]$. If we take $\alpha = \sum_{i=0}^{m} \alpha_i \frac{d^i}{dt^i} \in \tilde{\mathcal{D}}_{(q,u)}$, where $\alpha_i \in \tilde{\mathcal{M}}_{(q,u)}$, then $\frac{d}{dt}\alpha$ is defined as

$$\frac{d}{dt}\alpha := \sum_{i=0}^{m} \left( \alpha_i \frac{d^{i+1}}{dt^{i+1}} + \dot{\alpha}_i \frac{d^i}{dt^i} \right) \in \tilde{\mathcal{D}}_{(q,u)}.$$

Hence $\tilde{\mathcal{D}}_{(q,u)}$ is a left skew polynomial ring, and thus elements of $\tilde{\mathcal{D}}_{(q,u)}$ can act on the vector space $\tilde{\mathcal{E}}_{(q,u)}$ (see appendix C), that is, the vector space $\tilde{\mathcal{E}}_{(q,u)}$ can be endowed with a differential structure by defining a derivative operator $\frac{d}{dt}$ as follows:

$$\begin{aligned} & \frac{d}{dt} \sum_{l \geq 0} \left( \sum_{i=1}^{n} a_{i,l} dq_i^{(l)} + \sum_{k=1}^{m} b_{k,l} du_k^{(l)} \right) \\ := & \sum_{l \geq 0} \left( \sum_{i=1}^{n} \dot{a}_{i,l} dq_i^{(l)} + a_{k,l} dq_i^{(l+1)} + \sum_{k=1}^{m} \dot{b}_{k,l} du_k^{(l)} + b_{k,l} du_k^{(l+1)} \right), \end{aligned}$$

where $\sum_{l \geq 0} \left( \sum_{i=1}^{n} a_{i,l} dq_i^{(l)} + \sum_{k=1}^{m} b_{k,l} du_k^{(l)} \right) \in \tilde{\mathcal{E}}_{(q,u)}$. Furthermore, $\tilde{\mathcal{D}}_{(q,u)}$ is a non-commutative simple Euclidean domain (see proposition B.3 in appendix B).

Now, differentiating both sides of system (2.89), we have

$$\underbrace{\left( M(q) \frac{d^2}{dt^2} - \frac{\partial g}{\partial \dot{q}} \frac{d}{dt} + \left( \frac{\partial M}{\partial q_1} \ddot{q} \quad \cdots \quad \frac{\partial M}{\partial q_n} \ddot{q} \right) - \frac{\partial g}{\partial q} \quad -\frac{\partial g}{\partial u} \right)}_{\tilde{P}_{(q,u)}} \begin{pmatrix} dq \\ du \end{pmatrix} = 0.$$

Since each entry of $M(q)$ and $g(q, \dot{q}, u)$ are meromorphic with respect to each variable, $\tilde{P}_{(q,u)} \in \tilde{\mathcal{D}}_{(q,u)}^{n \times (n+m)}$. Since we have a similar proposition with proposition 2.1, we can define algebraic controllability of system (2.89) as follows.

**Definition 2.31** *System (2.89) is called* **algebraically controllable** *if $\tilde{P}_{(q,u)}$ is hyper-regular, that is, there exist unimodular matrices $\tilde{U}_{(q,u)} \in \tilde{\mathcal{D}}_{(q,u)}^{n\times n}$ and $\tilde{V}_{(q,u)} \in \tilde{\mathcal{D}}_{(q,u)}^{(n+m)\times(n+m)}$ such that*

$$\tilde{U}_{(q,u)}\tilde{P}_{(q,u)}\tilde{V}_{(q,u)} = \begin{pmatrix} I_n & 0 \end{pmatrix}.$$

From now on, we show that if system (2.89) is algebraically controllable, the transformed system expressed by first order differential equations is also algebraically controllable.

**Lemma 2.32** *System* (2.89) *is algebraically controllable if and only if system*

$$\ddot{q} = M^{-1}(q)g(q,\dot{q},u) \tag{2.90}$$

*is algebraically controllable.*

**Proof** From (2.90), we define $F := \ddot{q} - M^{-1}(q)g(q,\dot{q},u) = 0$. Then $dF = \hat{P}_{(q,u)}\begin{pmatrix} dq \\ du \end{pmatrix}$, where

$$\hat{P}_{(q,u)} := \begin{pmatrix} \frac{d^2}{dt^2}I_n - M^{-1}(q)\frac{\partial g}{\partial \dot{q}}\frac{d}{dt} - \frac{\partial}{\partial q}\left(M^{-1}(q)g\right) & -M^{-1}(q)\frac{\partial g}{\partial u} \end{pmatrix}.$$

Since

$$\frac{\partial M^{-1}}{\partial q_i} = -M^{-1}\frac{\partial M}{\partial q_i}M^{-1}, \quad i = 1, \cdots, n,$$

we have

$$\frac{\partial}{\partial q}\left(M^{-1}(q)g(q,\dot{q},u)\right) = -M^{-1}(q)\begin{pmatrix} \frac{\partial M}{\partial q_1}\ddot{q} & \cdots & \frac{\partial M}{\partial q_n}\ddot{q} \end{pmatrix} + M^{-1}(q)\frac{\partial g}{\partial q}.$$

Thus, we obtain

$$\hat{P}_{(q,u)} = M^{-1}(q)\tilde{P}_{(q,u)}.$$

Hence if system (2.89) is algebraically controllable, there exist unimodular matrices $\tilde{U}_{(q,u)}$, $\tilde{V}_{(q,u)}$ such that

$$\begin{aligned} &\tilde{U}_{(q,u)}\tilde{P}_{(q,u)}\tilde{V}_{(q,u)} = \begin{pmatrix} I_n & 0 \end{pmatrix} \\ \Leftrightarrow\ &(\tilde{U}_{(q,u)}M(q))(M^{-1}(q)\tilde{P}_{(q,u)})\tilde{V}_{(q,u)} = \begin{pmatrix} I_n & 0 \end{pmatrix} \\ \Leftrightarrow\ &(\tilde{U}_{(q,u)}M(q))\hat{P}_{(q,u)}\tilde{V}_{(q,u)} = \begin{pmatrix} I_n & 0 \end{pmatrix}. \end{aligned}$$

Therefore system (2.90) is also algebraically controllable.

Conversely, if system (2.90) is algebraically controllable, there exist unimodular matrices $\hat{U}_{(q,u)}$, $\hat{V}_{(q,u)}$ such that

$$\begin{aligned} & \hat{U}_{(q,u)}\hat{P}_{(q,u)}\hat{V}_{(q,u)} = \begin{pmatrix} I_n & 0 \end{pmatrix} \\ \Leftrightarrow\ & (\hat{U}_{(q,u)}M^{-1}(q))(M(q)\hat{P}_{(q,u)})\hat{V}_{(q,u)} = \begin{pmatrix} I_n & 0 \end{pmatrix} \\ \Leftrightarrow\ & (\hat{U}_{(q,u)}M^{-1}(q))\tilde{P}_{(q,u)}\hat{V}_{(q,u)} = \begin{pmatrix} I_n & 0 \end{pmatrix}. \end{aligned}$$

Therefore system (2.89) is also algebraically controllable. □

**Lemma 2.33** *System*

$$\ddot{q} = h(q, \dot{q}, u) \tag{2.91}$$

*is algebraically controllable if and only if system*

$$\begin{cases} \dot{q} = v, \\ \dot{v} = h(q, v, u) \end{cases} \tag{2.92}$$

*is algebraically controllable, where* $h : \boldsymbol{R}^n \times \boldsymbol{R}^n \times \boldsymbol{R}^m \to \boldsymbol{R}^n$ *is meromorphic with respect to each variable.*

**Proof** From (2.91), we define $F_1 := \ddot{q} - h(q, \dot{q}, u) = 0$. Then $dF_1 = P_1 \begin{pmatrix} dq \\ du \end{pmatrix}$, where

$$P_1 := \begin{pmatrix} \frac{d^2}{dt^2} I_n - \frac{\partial h}{\partial \dot{q}} \frac{d}{dt} - \frac{\partial h}{\partial q} & -\frac{\partial h}{\partial u} \end{pmatrix}.$$

In addition, from (2.92), we define $F_2 := \begin{pmatrix} \dot{q} - v \\ \dot{v} - h(q, v, u) \end{pmatrix} = 0$. Then $dF_2 = P_2 \begin{pmatrix} dq \\ dv \\ du \end{pmatrix}$, where

$$P_2 := \begin{pmatrix} \frac{d}{dt} I_n & -I_n & 0 \\ -\frac{\partial h}{\partial q} & \frac{d}{dt} I_n - \frac{\partial h}{\partial v} & -\frac{\partial h}{\partial u} \end{pmatrix}.$$

Then by a straightforward calculation,

$$P_3 := U_1 P_2 V_1 = \begin{pmatrix} P_1 & 0 \\ 0 & I_n \end{pmatrix}, \tag{2.93}$$

where $U_1 := \begin{pmatrix} \frac{d}{dt}I - \frac{\partial h}{\partial v} & I_n \\ I_n & 0 \end{pmatrix}$, $V_1 := \begin{pmatrix} I_n & 0 & 0 \\ \frac{d}{dt}I_n & 0 & -I_n \\ 0 & I_m & 0 \end{pmatrix}$. If system (2.91) is algebraically controllable, there exist unimodular matrices $U$, $V$ such that $UP_1V = \begin{pmatrix} I_n & 0 \end{pmatrix}$. Hence

$$\begin{pmatrix} U & 0 \\ 0 & I_n \end{pmatrix} P_3 \begin{pmatrix} V & 0 \\ 0 & I_n \end{pmatrix} = \begin{pmatrix} UP_1V & 0 \\ 0 & I_n \end{pmatrix} = \begin{pmatrix} I_n & 0 & 0 \\ 0 & 0 & I_n \end{pmatrix}.$$

Thus system (2.92) is algebraically controllable.

Conversely, suppose that system (2.92) is algebraically controllable. Since (2.93) is satisfied, if $P_3$ is hyper-regular, $P_1$ is also hyper-regular. Thus system (2.91) is algebraically controllable. □

Lemmas 2.32 and 2.33 yield the following theorem.

**Theorem 2.23** *System (2.89) is algebraically controllable if and only if system*

$$\begin{cases} \dot{q} = v, \\ \dot{v} = M^{-1}(q)g(q, v, u) \end{cases} \tag{2.94}$$

*is algebraically controllable.*

By theorem 2.23, if each entry of the matrix $M(q)$ is very complicated function, algebraic controllability of (2.94) can be examined by checking algebraic controllability of (2.89) without calculating $M^{-1}(q)$. However, many calculations might be required to directly check algebraic controllability of a given system (2.89). To reduce a computational complexity, in the next subsection, we show a more tractable condition for checking algebraic controllability of system (2.89).

### 2.7.1 Reduction condition for algebraic controllability

The section gives a reduction condition for checking algebraic controllability of mechanical control systems (2.89). To this end, we define

$$A(q, \dot{q}, \ddot{q}) := M(q)\ddot{q} - C(q, \dot{q}).$$

Then system (2.89) can be expressed by

$$E := A(q, \dot{q}, \ddot{q}) - B(q)u = 0.$$

First, let us split $q = \begin{pmatrix} q^1 \\ q^2 \end{pmatrix}$, $A(q,\dot{q},\ddot{q}) = \begin{pmatrix} A^1(q,\dot{q},\ddot{q}) \\ A^2(q,\dot{q},\ddot{q}) \end{pmatrix}$, and $B(q) = \begin{pmatrix} B^1(q) \\ B^2(q) \end{pmatrix}$, where

$$q^1 := \begin{pmatrix} q_1 \\ \vdots \\ q_{n-m} \end{pmatrix}, \quad q^2 := \begin{pmatrix} q_{n-m+1} \\ \vdots \\ q_n \end{pmatrix},$$

$$A^1(q,\dot{q},\ddot{q}) := \begin{pmatrix} A_1(q,\dot{q},\ddot{q}) \\ \vdots \\ A_{n-m}(q,\dot{q},\ddot{q}) \end{pmatrix}, \quad A^2(q,\dot{q},\ddot{q}) := \begin{pmatrix} A_{n-m+1}(q,\dot{q},\ddot{q}) \\ \vdots \\ A_n(q,\dot{q},\ddot{q}) \end{pmatrix},$$

$$B^1(q) := \begin{pmatrix} B_{1,1}(q) & \cdots & B_{1,m}(q) \\ \vdots & & \vdots \\ B_{(n-m),1}(q) & \cdots & B_{(n-m),m}(q) \end{pmatrix},$$

$$B^2(q) := \begin{pmatrix} B_{(n-m+1),1}(q) & \cdots & B_{(n-m+1),m}(q) \\ \vdots & & \vdots \\ B_{n,1}(q) & \cdots & B_{n,m}(q) \end{pmatrix}.$$

Then, differentiating $E = 0$, we get

$$dE = \underbrace{\begin{pmatrix} P_1^1 & P_2^1 & -B^1 \\ P_1^2 & P_2^2 & -B^2 \end{pmatrix}}_{P} \begin{pmatrix} dq^1 \\ dq^2 \\ du \end{pmatrix},$$

where

$$P_j^i := \frac{\partial A^i}{\partial q^j} + \frac{\partial A^i}{\partial \dot{q}^j}\frac{d}{dt} + \frac{\partial A^i}{\partial \ddot{q}^j}\frac{d^2}{dt^2} - \sum_{k=1}^{m} u_k \frac{\partial B_k^i}{\partial q^j},$$

and $B_k^i(q)$ represents $k$-th column vector of $B^i(q)$.

To get a sufficient condition for algebraic controllability, we put the following assumption.

**Assumption 2.1** *The matrix $B^2(q) \in \boldsymbol{R}^{m\times m}$ is invertible on $\boldsymbol{R}^n$ with some exceptional sets of measure zero.*

We note that for many practical systems, assumption 2.1 is satisfied because this assumption means that the number of independent control inputs equals $m$. Let us suppose that assumption 2.1 holds. Then,

$$PV_1 = \begin{pmatrix} P_1^1 & P_2^1 & B^1(B^2)^{-1} \\ P_1^2 & P_2^2 & I_m \end{pmatrix},$$

where

$$V_1 := \begin{pmatrix} I_{n-m} & & \\ & I_m & \\ & & -(B^2)^{-1} \end{pmatrix}.$$

In addition,

$$\begin{aligned} \tilde{P} &:= U_1 P V_1 V_2 V_3 \\ &= \begin{pmatrix} P_1^1 - B^1(B^2)^{-1}P_1^2 & P_2^1 - B^1(B^2)^{-1}P_2^2 & 0 \\ 0 & 0 & I_m \end{pmatrix}, \end{aligned}$$

where

$$U_1 := \begin{pmatrix} I_{n-m} & -B^1(B^2)^{-1} \\ & I_m \end{pmatrix},\ V_2 := \begin{pmatrix} I_{n-m} & & \\ & I_m & \\ -P_1^2 & & I_m \end{pmatrix},\ V_3 := \begin{pmatrix} I_{n-m} & & \\ & I_m & \\ & -P_2^2 & I_m \end{pmatrix}.$$

We can conclude that if assumption 2.1 and the following assumption hold, then system (2.89) is algebraically controllable.

**Assumption 2.2** *The matrix $P_1^1 - B^1(B^2)^{-1}P_1^2 \in \tilde{\mathcal{D}}_{(q,u)}^{(n-m)\times(n-m)}$ is unimodular or the matrix $P_2^1 - B^1(B^2)^{-1}P_2^2 \in \tilde{\mathcal{D}}_{(q,u)}^{(n-m)\times m}$ is hyper-regular.*

From now on, we prove the above mentioned fact.

**Case 1: $P_1^1 - B^1(B^2)^{-1}P_1^2$ is unimodular**

Since $P_1^1 - B^1(B^2)^{-1}P_1^2 \in \tilde{\mathcal{D}}_{(q,u)}^{(n-m)\times(n-m)}$ is unimodular, there exists unimodular matrix $R_1 \in \tilde{\mathcal{D}}_{(q,u)}^{(n-m)\times(n-m)}$ such that

$$(P_1^1 - B^1(B^2)^{-1}P_1^2)R_1 = I_{n-m}.$$

Correspondingly, we have

$$\tilde{P}V_4 = \begin{pmatrix} I_{n-m} & P_2^1 - B^1(B^2)^{-1}P_2^2 & 0 \\ 0 & 0 & I_m \end{pmatrix},$$

where

$$V_4 := \begin{pmatrix} R_1 & & \\ & I_m & \\ & & I_m \end{pmatrix}.$$

Hence

$$U_2\tilde{P}V_4V_5V_6 = \begin{pmatrix} I_n & 0 \end{pmatrix}, \tag{2.95}$$

where

$$V_5 := \begin{pmatrix} I_{n-m} & -(P_2^1 - B^1(B^2)^{-1}P_2^2) & \\ & I_m & \\ & & I_m \end{pmatrix},$$

$$V_6 := \begin{pmatrix} I_{n-m} & & \\ & 0 & I_m \\ & I_m & 0 \end{pmatrix}.$$

From (2.95), if assumption 2.1 holds and $P_1^1 - B^1(B^2)^{-1}P_1^2$ is unimodular, system (2.89) is algebraically controllable.

**Case 2: $P_2^1 - B^1(B^2)^{-1}P_2^2$ is hyper-regular**

Since $P_2^1 - B^1(B^2)^{-1}P_2^2 \in \tilde{\mathcal{D}}_{(q,u)}^{(n-m)\times m}$ is hyper-regular, $n \leq 2m$ and there exist unimoudular matrices $L_1 \in \tilde{\mathcal{D}}_{(q,u)}^{(n-m)\times(n-m)}$ and $R_2 \in \tilde{\mathcal{D}}_{(q,u)}^{m\times m}$ such that

$$L_1(P_2^1 - B^1(B^2)^{-1}P_2^2)R_2 = \begin{pmatrix} I_{n-m} & 0 \end{pmatrix}.$$

Correspondingly, we have

$$U_2\tilde{P}V_7 = \begin{pmatrix} L_1(P_1^1 - B^1(B^2)^{-1}P_1^2) & I_{n-m} & 0 & 0 \\ 0 & 0 & 0 & I_m \end{pmatrix},$$

where

$$U_2 := \begin{pmatrix} L_1 & \\ & I_m \end{pmatrix}, \; V_7 := \begin{pmatrix} I_{n-m} & & \\ & R_2 & \\ & & I_m \end{pmatrix}.$$

Hence

$$U_2\tilde{P}(V_7 \cdots V_{10}) = \begin{pmatrix} I_n & 0 \end{pmatrix}, \tag{2.96}$$

where

$$V_8 := \begin{pmatrix} I_{n-m} & & & \\ -L_1(P_1^1 - B^1(B^2)^{-1}P_1^2) & I_{n-m} & 0 & \\ 0 & 0 & I_{2m-n} & \\ & & & I_m \end{pmatrix},$$

$$V_9 := \begin{pmatrix} 0 & I_{n-m} & 0 & \\ I_{n-m} & 0 & 0 & \\ 0 & 0 & I_{2m-n} & \\ & & & I_m \end{pmatrix}, \; V_{10} := \begin{pmatrix} I_{n-m} & & \\ & 0 & I_m \\ & I_m & 0 \end{pmatrix}.$$

From (2.96), if assumption 2.1 holds and $P_2^1 - B^1(B^2)^{-1}P_2^2$ is hyper-regular, system (2.89) is algebraically controllable.

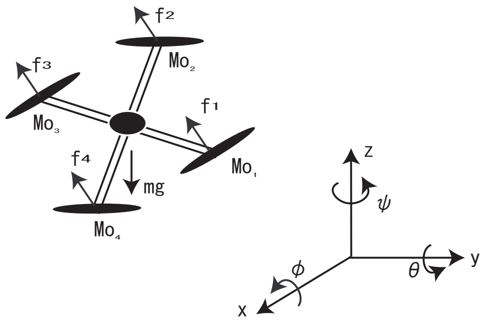



Figure 2.2: Quadrotor UAV.

### 2.7.2 Quadrotor unmanned aerial vehicle

Using a quadrotor unmanned aerial vehicle example [49], which is a mechanical control system with six degrees of freedom and four control inputs, we demonstrate that assumptions 2.1 and 2.2 reduce a computational complexity for checking whether or not a given system (2.89) is algebraically controllable, that is, accessible.

We regard the quadrotor UAV as a rigid body, whose configuration space is $\boldsymbol{R}^3 \times SO(3)$ [77]. Since $\boldsymbol{R}^3 \times SO(3)$ is a six dimensional manifold, we can locally consider $\boldsymbol{R}^3 \times SO(3)$ as $\boldsymbol{R}^6$. Let $(x, y, z, \phi, \theta, \psi)$ be local coordinates of $\boldsymbol{R}^3 \times SO(3)$, where $(x, y, z)$ denotes the position of the center of gravity of the quadrotor UAV, and $\phi$, $\theta$, and $\psi$ denote the roll, pitch, and yaw angles of UAV in an inertial frame, respectively. The Lagrangian of this system $L : T(\boldsymbol{R}^3 \times SO(3)) \to \boldsymbol{R}$ is given by $L := \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \frac{1}{2}\omega^T \begin{pmatrix} J_1 & 0 & 0 \\ 0 & J_2 & 0 \\ 0 & 0 & J_3 \end{pmatrix} \omega - mgz$, where $m$ denotes the mass of the vehicle and $g$ is the gravitational acceleration. Further, $\omega$ denotes the angular velocity of the vehicle in the body frame [77], and is expressed as $\omega = \begin{pmatrix} 1 & 0 & -\sin\theta \\ 0 & \cos\phi & \sin\phi\cos\theta \\ 0 & -\sin\phi & \cos\phi\cos\theta \end{pmatrix} \begin{pmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix}$. In terms of the local coordinates $q := (x, y, z, \phi, \theta, \psi)$, the Lagrangian control system of the quadrotor UAV is then subject to the equations of motion

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = B(q)u \quad \Leftrightarrow \quad M(q)\ddot{q} = C(q, \dot{q}) + B(q)u, \tag{2.97}$$

where $u := (u_1, \cdots, u_4)$ and

$$M(q) := \begin{pmatrix} m & 0 & 0 & 0 & 0 & 0 \\ 0 & m & 0 & 0 & 0 & 0 \\ 0 & 0 & m & 0 & 0 & 0 \\ 0 & 0 & 0 & J_1 & 0 & -J_1 \sin\theta \\ 0 & 0 & 0 & 0 & J_2 \cos^2\phi + J_3 \sin^2\phi & m_1 \\ 0 & 0 & 0 & -J_1 \sin\theta & m_1 & m_2 \end{pmatrix},$$

$$C(q, \dot{q}) := \begin{pmatrix} 0 \\ 0 \\ -mg \\ c_1 \\ c_2 \\ c_3 \end{pmatrix},$$

$$B(q) := \begin{pmatrix} \cos\phi \sin\theta \cos\psi + \sin\phi \sin\psi & 0 & 0 & 0 \\ \cos\phi \sin\theta \sin\psi - \sin\phi \cos\psi & 0 & 0 & 0 \\ \cos\phi \cos\theta & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\phi & \sin\phi \\ 0 & -\sin\theta & \sin\phi \cos\theta & \cos\phi \cos\theta \end{pmatrix},$$

and where

$$\begin{aligned} m_1 &:= (J_2 - J_3) \sin\phi \cos\phi \cos\theta, \\ m_2 &:= J_1 \sin^2\theta + (J_2 \sin^2\phi + J_3 \cos^2\phi) \cos^2\theta, \\ c_1 &:= J_1 \cos\theta \dot{\theta} \dot{\psi} - (J_2 - J_3) \sin\phi \cos\phi (\dot{\theta}^2 - \dot{\psi}^2 \cos^2\theta - (\sin^2\phi - \cos^2\phi) \cos\theta \dot{\theta} \dot{\psi}, \\ c_2 &:= -J_1 \cos\theta (\dot{\phi} - \dot{\psi} \sin\theta) \dot{\psi} - (J_2 - J_3) \Big( (\cos^2\phi - \sin^2\phi) \cos\theta \dot{\phi} \dot{\psi} \\ &\qquad + \sin\phi \cos\phi \dot{\phi} \dot{\theta} \Big) - (J_2 \sin^2\phi + J_3 \cos^2\phi) \sin\theta \cos\theta \dot{\psi}^2, \\ c_3 &:= \big( J_1 - (J_2 - J_3)(\cos^2\phi - \sin^2\phi) \big) \cos\theta \cdot \dot{\phi} \dot{\theta} \\ &\qquad - 2 \big( J_1 - (J_2 \sin^2\phi + J_3 \cos^2\phi) \big) \sin\theta \cos\theta \cdot \dot{\theta} \dot{\psi} \\ &\qquad (J_2 - J_3) \cos\phi \sin\phi \left( \sin\theta \cdot \dot{\theta}^2 - \cos^2\theta \cdot \dot{\phi} \dot{\psi} \right). \end{aligned}$$

Here, $u_1$ is the total thrust produced by the four rotors $\mathrm{Mo}_i$, $i = 1, \cdots, 4$, that is, it is given by $u_1 := f_1 + f_2 + f_3 + f_4$, where $f_i := k_i w_i^2$ is the thrust generated by $\mathrm{Mo}_i$ and $k_i > 0$ is a constant, and $w_i$ is the angular speed of $\mathrm{Mo}_i$. The control inputs $u_2$, $u_3$, and $u_4$ are the generalized moments; they are given by $u_2 := (f_3 - f_1)h$, $u_3 := (f_2 - f_4)h$ and $u_4 := (f_2 + f_4 - f_1 - f_3)\kappa$, where $h$ represents the distance from each $\mathrm{Mo}_i$ to the center of gravity of the quadrotor UAV and $\kappa$ is a constant.

From Eq. (2.97) let $E := M(q)\ddot{q} - C(q, \dot{q}) - B(q)u = 0$. Differentiating $E$, we get

$$dE = P \begin{pmatrix} dq^1 \\ dq^2 \\ du \end{pmatrix},$$

where $q^1 := (x, y)$, $q^2 := (z, \phi, \theta, \psi)$, and

$$P := \begin{pmatrix} P_1^1 & P_2^1 & -B^1 \\ 0 & P_2^2 & -B^2 \end{pmatrix},$$

and where

$$P_1^1 := \begin{pmatrix} m\frac{d^2}{dt^2} & 0 \\ 0 & m\frac{d^2}{dt^2} \end{pmatrix}, \ P_2^1 := \begin{pmatrix} 0 & a_1 & a_2 & a_3 \\ 0 & a_4 & a_5 & a_6 \end{pmatrix},$$

$$P_2^2 := \begin{pmatrix} m\frac{d^2}{dt^2} & u_1 \sin\phi\cos\theta & u_1\cos\phi\sin\theta & 0 \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{pmatrix},$$

$$B^1 := \begin{pmatrix} \cos\phi\sin\theta\cos\psi + \sin\phi\sin\psi & 0 & 0 & 0 \\ \cos\phi\sin\theta\sin\psi - \sin\phi\cos\psi & 0 & 0 & 0 \end{pmatrix},$$

$$B^2 := \begin{pmatrix} \cos\phi\cos\theta & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\phi & -\sin\phi \\ 0 & -\sin\theta & \cos\theta\sin\phi & \cos\phi\cos\theta \end{pmatrix},$$

where $a_1 := u_1 (\sin\phi\sin\theta\cos\psi - \cos\phi\sin\psi)$, $a_2 := -u_1\cos\phi\cos\theta\cos\psi$, $a_3 := u_1 (\cos\phi\sin\theta\sin\psi - \sin\phi\cos\psi)$, $a_4 := u_1 (\sin\phi\sin\theta\sin\psi + \cos\phi\cos\psi)$, $a_5 := -u_1\cos\phi\cos\theta\sin\psi$, $a_6 := -u_1 (\cos\phi\sin\theta\cos\psi + \sin\phi\sin\psi)$, and $*$ are omitted because they are not necessary in later calculations. If we check whether or not system (2.97) is algebraically controllable, many calculations are required because matrix size of $P$ is $6 \times 10$. In order to reduce a computational complexity, we should check whether or not assumptions 2.1 and 2.2 hold because if assumptions 2.1 and 2.2 hold, system (2.97) is algebraically controllable.

Since $\det B^2 = \cos\phi\cos^2\theta$, assumption 2.1 holds. By a direct calculation,

$$(B^2)^{-1} = \begin{pmatrix} \frac{1}{\cos(\phi)\cos(\theta)} & 0 & 0 & 0 \\ 0 & * & 0 & 0 \\ 0 & * & * & * \\ 0 & * & * & * \end{pmatrix}.$$

Next, let us check whether or not assumption 2.2 holds. Clearly, $P_1^1$ is not unimodular. Thus we check whether or not $P_2^1 - B^1(B^2)^{-1}P_2^2$ is hyper-regular.

Now we put $P_2^1 - B^1(B^2)^{-1}P_2^2 = \begin{pmatrix} p_1 & p_2 & p_3 & p_4 \\ p_5 & p_6 & p_7 & p_8 \end{pmatrix}$, where $p_4 = a_3$, $p_8 = a_6$. To check whether or not $P_2^1 - B^1(B^2)^{-1}P_2^2$ is hyper-regular, we repeat elementary column operations. Correspondingly, we get the following unimodular matrices.

$$V_1 := \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & \frac{1}{p_8} \end{pmatrix}, \; V_2 := \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & -p_8 & & 1 \end{pmatrix}, \; V_3 := \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & -p_7 & 1 \end{pmatrix},$$

$$V_4 := \begin{pmatrix} 1 & & & \\ & \frac{p_8}{p_3p_8 - p_4p_7} & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \; V_5 := \begin{pmatrix} 1 & & & \\ & 1 & & \\ & -p_2 + \frac{p_4}{p_8}p_6 & 1 & \\ & & & 1 \end{pmatrix},$$

$$V_6 := \begin{pmatrix} 1 & & & \\ & 1 & & -\frac{p_4}{p_8} \\ & & 1 & \\ & & & 1 \end{pmatrix}, \; V_7 := \begin{pmatrix} 1 & & & \\ & 1 & & \\ -p_1 & & 1 & \\ & & & 1 \end{pmatrix}, \; V_8 := \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ -p_5 & & & 1 \end{pmatrix},$$

$$V_9 := \begin{pmatrix} 0 & & 1 & \\ & 1 & & \\ 1 & & 0 & \\ & & & 1 \end{pmatrix}, \; V_{10} := \begin{pmatrix} 1 & & & \\ & 0 & & 1 \\ & & 1 & \\ & 1 & & 0 \end{pmatrix},$$

where $p_8 = a_6 = -m\ddot{x}$ and $p_3p_8 - p_4p_7 = u_1^2 \tan\theta$. Then we have

$$\left(P_2^1 - B^1(B^2)^{-1}P_2^2\right)(V_1 \cdots V_{10}) = \begin{pmatrix} I_2 & 0 \end{pmatrix}.$$

Hence $P_2^1 - B^1(B^2)^{-1}P_2^2$ is hyper-regular and assumption 2.2 holds. Thus there exist unimodular matrices $U$ and $V$ such that

$$UPV = \begin{pmatrix} I_6 & 0 \end{pmatrix}.$$

Therefore system (2.97) is algebraically controllable, that is, accessible.

## 2.8 Trajectory tracking control of non-algebraically controllable systems

This section considers a trajectory tracking control of non-algebraically controllable affine system

$$\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x)u_i, \tag{2.98}$$

where $f, g_i : \boldsymbol{R}^n \to \boldsymbol{R}^n$, $1 \leq i \leq m$ are meromorphic with respect to each variable. In particular, for simplicity, let us consider the following special system of the affine nonlinear system

$$\dot{x}^1 = f^1(x^1, x^2) + g^1(x^1, x^2)u, \tag{2.99}$$
$$\dot{x}^2 = f^2(x^2), \tag{2.100}$$

where $x^1 := (x_1, \cdots, x_{n_1}) \in \boldsymbol{R}^{n_1}$ and $x^2 := (x_{n_1+1}, \cdots, n_{n_1+n_2}) \in \boldsymbol{R}^{n_2}$ denote state variables, and $u \in \boldsymbol{R}^m$ denotes an input variable. Moreover, $f^1 : \boldsymbol{R}^{n_1} \times \boldsymbol{R}^{n_2} \to \boldsymbol{R}^{n_1}$, $f^2 : \boldsymbol{R}^{n_2} \to \boldsymbol{R}^{n_2}$, and $g_1 : \boldsymbol{R}^{n_1} \times \boldsymbol{R}^{n_2} \to \boldsymbol{R}^{n \times m}$ are meromorphic with respect to each variable. Fig. 2.3 illustrates the structure of system (2.99)-(2.100).

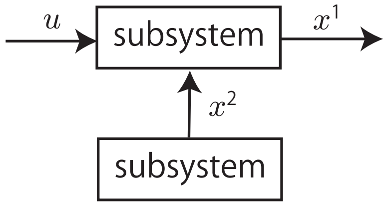



Figure 2.3: Structure of system (2.99)-(2.100)

**Remark 2.11** *Let us consider linear time invariant system*

$$\dot{x} = Ax + Bu, \tag{2.101}$$

*where $x \in \boldsymbol{R}^n$ and $u \in \boldsymbol{R}^m$ Now assume that (i) there exists a d-dimensional subspace $V$ of $\boldsymbol{R}^n$ such that $V$ is invariant under $A$.*

*After a change of coordinates, without loss of generality, we can suppose that*

$$V = \operatorname{span}\left\{(v_1, \cdots, v_d, 0, \cdots, 0),\ v_i \in \boldsymbol{R},\ i = 1, \cdots, d\right\}.$$

*Then because of the invariance of $V$ under $A$, the matrix $A$ has a block triangular structure*

$$A = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}.$$

*Moreover, suppose that (ii) $Bu \in V$ for all $u \in \boldsymbol{R}^m$. Then after the same change of coordinates, the matrix $B$ forms*

$$B = \begin{pmatrix} B_1 \\ 0 \end{pmatrix}.$$

*Therefore if there exists a subspace $V$ satisfying (i) and (ii), after a change of coordinates, linear system* (2.101) *can be decomposed into*

$$\dot{x}^1 = A_{11}x^1 + A_{12}x^2 + B_1 u, \tag{2.102}$$

$$\dot{x}^2 = A_{22}x^2. \tag{2.103}$$

*The structure of linear system* (2.102)-(2.103) *is the same form as* (2.99)-(2.100).

*Similarly, it is known [36, 82] that under certain assumptions, affine nonlinear system* (2.98) *can be decomposed into* (2.99)-(2.100). ■

Since the variable $x^2$ is not influenced by a control input, a differential $d\phi(x^2)$ of a meromorphic function $\phi(x^2)$ has infinite relative degree. Hence by proposition 2.17, $d\phi(x^2)$ is an autonomous variable and system (2.99)-(2.100) is not accessible. Therefore by corollary 2.21, system (2.99)-(2.100) is not algebraically controllable.

However, a trajectory tracking control of total system (2.99)-(2.100) can be realized as shown in the following example.

**Example 2.11** Let us consider example 2.7, again. From example 2.7, system (2.42)-(2.43) is not algebraically controllable.

However, if we apply a feedback $u = -x_1x_2 + v$ into subsystem (2.42), subsystem (2.42) is transformed into

$$\dot{x}_1 = v,$$

where $v$ is a new input variable. Furthermore, Eq. (2.43) implies that

$$x_2(t) = x_2(0)\exp(-t).$$

Hence if we consider $(x_1(t), x_2(t)) = (x_1^*(t), 0)$ as a reference trajectory of system (2.42)-(2.43), and if we apply $v = \dot{x}_1^*(t) - k(x_1 - x_1^*(t))$, $k > 0$, that is, $u = -x_1x_2 + (\dot{x}_1^*(t) - k(x_1 - x_1^*(t)))$ into system (2.42)-(2.43), the actual trajectory $(x_1(t), x_2(t))$ asymptotically approaches the reference trajectory $(x_1^*(t), 0)$. ■

In the above example, we have used exact feedback linearization method [36, 82] (see appendix F). If subsystem (2.99) can be transformed into a linear system using exact feedback linearization method and if $x_2 = 0$ of subsystem (2.100) is asymptotically stable, it is possible to design a controller such that the actual trajectory $(x^1(t), x^2(t))$ asymptotically approaches $(x^{1*}(t), 0)$, where $x^{1*}(t)$ is an appropriate reference trajectory of $x^1(t)$.

Even if we cannot use exact feedback linearization method, it is possible to realize a trajectory tracking control based on linear approximation method along a reference trajectory $(x_1^*(t), 0)$ as follows.

$$\begin{pmatrix} \dot{x}_\epsilon^1 \\ \dot{x}_\epsilon^2 \end{pmatrix} = \begin{pmatrix} A_{11}(t) & A_{12}(t) \\ 0 & A_{22}(t) \end{pmatrix} \begin{pmatrix} x_\epsilon^1 \\ x_\epsilon^2 \end{pmatrix} + \begin{pmatrix} B_1(t) \\ 0 \end{pmatrix} u_\epsilon, \tag{2.104}$$

where

$$\begin{aligned}
A_{11}(t) &:= \frac{\partial f^1}{\partial x^1}(x^{1*}(t), 0) + \sum_{i=1}^{m} \frac{\partial g_j^1}{\partial x^1}(x^{1*}(t), 0) u_j^*(t), \\
A_{12}(t) &:= \frac{\partial f^1}{\partial x^2}(x^{1*}(t), 0) + \sum_{i=1}^{m} \frac{\partial g_j^1}{\partial x^2}(x^{1*}(t), 0) u_j^*(t), \\
A_{22}(t) &:= \frac{\partial f^2}{\partial x^2}(x^{1*}(t), 0), \\
B_1(t) &:= g^1(x^{1*}(t), 0).
\end{aligned}$$

Although linear system (2.104) is not completely controllable, if the system is exponentially stabilizable, it is possible to design a controller such that the actual trajectory $(x^1(t), x^2(t))$ asymptotically approaches $(x^{1*}(t), 0)$.

**Remark 2.12** *It is an open problem that "What is a class of* **non**-*algebraically controllable systems* (2.99)-(2.100) *whose linearizations* (2.104) *along trajectories* $(x^{1*}(t), 0)$ *are exponentially* **stabilizable**? *" However, we note that in [26], it has been shown one theoretical limit to the tracking performance that can be obtained in systems with zero dynamics (see appendix F).* ■

## 2.9 Summary

This chapter has clarified a class of nonlinear systems described by ordinary differential equations such that trajectory tracking controls are easily realized. First, we have introduced algebraic controllability and controllable trajectory in order to give a class of nonlinear systems whose linearizations are uniformly completely controllable. Next, we have introduced algebraic observability and observable trajectory in order to give a class of nonlinear systems whose linearizations are uniformly completely observable. We have also explained that the concepts of controllable trajectory and observable trajectory are needed only for nonlinear systems. Furthermore, we have shown that if a given system is algebraically controllable and observable, LQ optimal control method is useful to design a feedback controller such that the actual trajectory asymptotically approaches a periodic reference trajectory. Moreover, we have proven that the concepts of algebraic controllability and accessibility are equivalent, and for nonlinear mechanical control systems, we have given a reduction condition for checking whether or not the system is algebraically controllable. Finally, we have considered a trajectory tracking control of non-algebraically controllable affine system.

# Chapter 3

# Differential algebraic systems

Differential algebraic systems (DAS) arise naturally as dynamical model of electrical [95], mechanical [59], and chemical engineering [60] applications. The chapter studies DAS with geometric index one. Similarly to the case of nonlinear systems described by ordinary differential equations, we have the following questions.

- What is a class of DAS whose linearizations along trajectories are uniformly completely controllable (observable)?

- What is a class of trajectories stated in the above question?

If we can answer the above questions, we get a class of DAS and reference trajectories such that trajectory tracking controls are easily realized. In order to answer the questions, the chapter also introduces algebraic controllability and algebraic observability for DAS with geometric index one, and introduces controllable trajectory and observable trajectory. Similarly to the case of nonlinear systems expressed by ordinary differential equations, it is shown that if a given nonlinear DAS with geometric index one is algebraically controllable, every linearized system along any (periodic) controllable trajectory is (uniformly) completely controllable. As a dual result, it is shown that if a given nonlinear DAS with geometric index one is algebraically observable, every linearized system along any (periodic) observable trajectory is (uniformly) completely observable.

Incidentally, when we study DAS, a choice of independent input variables might be not obvious because the system is constrained by algebraic equations. For example, according to [17], when Y or $\Delta$ connections are used in a three-phase permanent-magnet synchronous machine (PMSM), a choice of independent input variables is not obvious because the system is constrained by the Kirchhoff's law. Hence it is meaningful not to split up into state and input variables. In section 3.3, we study differential flatness of DAS which does not distinguish state, input, and output variables.

## 3.1 DAS with geometric index one

In this chapter, we study a trajectory tracking control of the following system.

$$\dot{x} = f(x, \tilde{x}, u), \tag{3.1}$$

$$0 = g(x, \tilde{x}), \tag{3.2}$$

$$y = h(x, \tilde{x}), \tag{3.3}$$

where $(x, \tilde{x}) \in \boldsymbol{R}^n \times \boldsymbol{R}^{\tilde{n}}$, $u \in \boldsymbol{R}^m$, and $y \in \boldsymbol{R}^p$ denote state, input, and output variables, respectively. Moreover, $f : \boldsymbol{R}^n \times \boldsymbol{R}^{\tilde{n}} \times \boldsymbol{R}^m \to \boldsymbol{R}^n$, $g : \boldsymbol{R}^n \times \boldsymbol{R}^{\tilde{n}} \to \boldsymbol{R}^{\tilde{n}}$, and $h : \boldsymbol{R}^n \times \boldsymbol{R}^{\tilde{n}} \to \boldsymbol{R}^p$ are meromorphic on $\boldsymbol{R}^n \times \boldsymbol{R}^{\tilde{n}} \times \boldsymbol{R}^m$, $\boldsymbol{R}^n \times \boldsymbol{R}^{\tilde{n}}$, and $\boldsymbol{R}^n \times \boldsymbol{R}^{\tilde{n}}$, respectively. Let

$$W := \left\{ (x, \tilde{x}) \in \boldsymbol{R}^n \times \boldsymbol{R}^{\tilde{n}} \,\middle|\, g(x, \tilde{x}) = 0, \det \frac{\partial g}{\partial \tilde{x}}(x, \tilde{x}) \neq 0 \right\}.$$

System (3.1)-(3.3) is called a differential algebraic system with **geometric index one** if $W \neq \emptyset$. More precisely, see [91, 95].

**Remark 3.1** *Differential algebraic systems are also called* **descriptor systems** *[69, 70]* ■

**Remark 3.2** *In addition to geometric index, there are some concepts of index such as* **differentiation index** *[95]. Roughly speaking, the differentiation index is defined as the number of differentiations with respect to $t$ needed to express $\tilde{x}$ in terms of $x$ [7]. If we use the differentiation index framework, it is difficult to choose appropriate initial conditions more than in the case of geometric index framework.*

*For example, let us consider*

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -x_3 \\ x_3 \end{pmatrix} \tag{3.4}$$

$$0 = x_3 + x_2 - x_1. \tag{3.5}$$

**Case 1: Geometric index framework**

*Clearly, system* (3.4)-(3.5) *has geometric index one. Substituting $x_3 = x_1 - x_2$ deduced from* (3.5) *into* (3.4)*, we have*

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

*Integrating the above equation, we obtain*

$$\begin{aligned} x_1(t) &= \frac{1}{2}(x_1(0) + x_2(0)) + \frac{1}{2}(x_1(0) - x_2(0)) \exp(-2t), \\ x_2(t) &= \frac{1}{2}(x_1(0) + x_2(0)) - \frac{1}{2}(x_1(0) - x_2(0)) \exp(-2t), \\ x_3(t) &= (x_1(0) - x_2(0)) \exp(-2t). \end{aligned}$$

*Since* $x_3(t) + x_2(t) - x_1(t) = 0$ *for all* $t \in \boldsymbol{R}$, *the above solution always satisfies* (3.5).

**Case 2: Differentiation index framework**

*Differentiating* (3.5) *with respect to* $t$, *we get the following equation*

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 1 & 1 \end{pmatrix} \frac{d}{dt} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -x_3 \\ x_3 \\ 0 \end{pmatrix} \Leftrightarrow \frac{d}{dt} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -x_3 \\ x_3 \\ 0 \end{pmatrix} \tag{3.6}$$

*Integrating the above equation, we obtain*

$$\begin{aligned} x_1(t) &= -x_3(0)t + x_1(0), \\ x_2(t) &= x_3(0)t + x_2(0), \\ x_3(t) &= x_3(0). \end{aligned}$$

*Since* $x_3(t) + x_2(t) - x_1(t) = x_3(0) + x_1(0) - x_2(0) + 2x_3(0)t$, *if*

$$x_1(0) = x_2(0), \;\; x_3(0) = 0, \tag{3.7}$$

*algebraic constraint* (3.5) *is alway satisfied. However if* (3.7) *is not satisfied,* (3.5) *is not always satisfied.*

*We note that in contrast to the underlying ODE arising in the differentiation index framework, the state dimension of the resulting ODE in the geometric index framework is strictly lower than that of the original DAE.* ■

First, we define trajectory of system (3.1)-(3.3).

**Definition 3.1** *A trajectory of system (3.1)-(3.3) is a triple* $(x^*(t), \tilde{x}^*(t), u^*(t))$ *satisfying*

$$\begin{cases} \dot{x}^*(t) = f(x^*(t), \tilde{x}^*(t), u^*(t)), \\ 0 = g(x^*(t), \tilde{x}^*(t)) \end{cases} \quad \text{for almost all} \quad t \in \boldsymbol{R}.$$

*A trajectory* $(x^*(t), \tilde{x}^*(t), u^*(t))$ *of system* (3.1)-(3.3) *is called* **periodic** *if* $x_i^*(t)$, $\tilde{x}_j^*(t)$, *and* $u_k^*(t)$, $1 \le i \le n$, $1 \le j \le \tilde{n}$, $1 \le k \le m$ *are periodic with the same period.*

Let $(x^*(t), \tilde{x}^*(t), u^*(t))$ be a trajectory for system (3.1)-(3.3), and let $(x^*(t), \tilde{x}^*(t))$ be a reference trajectory for system (3.1)-(3.3). Moreover suppose that $(x^*(t), \tilde{x}^*(t)) \in W$ on $\boldsymbol{R}$. Then we can analyze error dynamics between the actual and reference trajectories as follows. Let $x_\epsilon := x - x^*$, $\tilde{x}_\epsilon := \tilde{x} - \tilde{x}^*$, $u_\epsilon := u - u^*$, and $y_\epsilon := y - y^*$. Then we have

$$\begin{cases} \dot{x}_\epsilon = f(x_\epsilon + x^*(t), \tilde{x}_\epsilon + \tilde{x}^*(t), u_\epsilon + u^*(t)) - f(x^*(t), \tilde{x}^*(t), u^*(t)), \\ 0 = g(x_\epsilon + x^*(t), \tilde{x}_\epsilon + \tilde{x}^*(t)), \\ y_\epsilon = h(x_\epsilon + x^*(t), \tilde{x}_\epsilon + \tilde{x}^*(t)) - h(x^*(t), \tilde{x}^*(t)). \end{cases} \tag{3.8}$$

Linearizing system (3.8) at $x_\epsilon = 0$, $\tilde{x}_\epsilon = 0$, and $u_\epsilon = 0$, we have

$$\begin{cases} \dot{x}_\epsilon = \frac{\partial f}{\partial x}(x^*(t), \tilde{x}^*(t), u^*(t))x_\epsilon + \frac{\partial f}{\partial \tilde{x}}(x^*(t), \tilde{x}^*(t), u^*(t))\tilde{x}_\epsilon + \frac{\partial f}{\partial u}(x^*(t), \tilde{x}^*(t), u^*(t))u_\epsilon, \\ 0 = g(x^*(t), \tilde{x}^*(t)) + \frac{\partial g}{\partial x}(x^*(t), \tilde{x}^*(t))x_\epsilon + \frac{\partial g}{\partial \tilde{x}}(x^*(t), \tilde{x}^*(t))\tilde{x}_\epsilon, \\ y_\epsilon = \frac{\partial h}{\partial x}(x^*(t), \tilde{x}^*(t))x_\epsilon + \frac{\partial h}{\partial \tilde{x}}(x^*(t), \tilde{x}^*(t))\tilde{x}_\epsilon \end{cases} \tag{3.9}$$

Since $(x^*(t), \tilde{x}^*(t), u^*(t))$ is a trajectory of system (3.1)-(3.3) and $(x^*(t), \tilde{x}^*(t)) \in W$ on $\boldsymbol{R}$, $g(x^*(t), \tilde{x}^*(t)) = 0$ and the matrix $\frac{\partial g}{\partial \tilde{x}}(x^*(t), \tilde{x}^*(t))$ is invertible on $\boldsymbol{R}$. Hence (3.9) is equivalent to

$$\begin{cases} \dot{x}_\epsilon = A(t)x_\epsilon + B(t)u_\epsilon, \\ y_\epsilon = C(t)x_\epsilon, \end{cases} \tag{3.10}$$

where

$$\begin{aligned} A(t) &:= \frac{\partial f}{\partial x}(x^*(t), \tilde{x}^*(t), u^*(t)) - \frac{\partial f}{\partial \tilde{x}}(x^*(t), \tilde{x}^*(t), u^*(t)) \\ &\quad \times \left(\frac{\partial g}{\partial \tilde{x}}(x^*(t), \tilde{x}^*(t))\right)^{-1} \frac{\partial g}{\partial x}(x^*(t), \tilde{x}^*(t)), && (3.11) \\ B(t) &:= \frac{\partial f}{\partial u}(x^*(t), \tilde{x}^*(t), u^*(t)), && (3.12) \\ C(t) &:= \frac{\partial h}{\partial x}(x^*(t), \tilde{x}^*(t)) - \frac{\partial h}{\partial \tilde{x}}(x^*(t), \tilde{x}^*(t)) \\ &\quad \times \left(\frac{\partial g}{\partial \tilde{x}}(x^*(t), \tilde{x}^*(t))\right)^{-1} \frac{\partial g}{\partial x}(x^*(t), \tilde{x}^*(t)). && (3.13) \end{aligned}$$

Hence if we design a feedback controller of system (3.10) such that the origin is exponentially stable, by applying the same controller into system (3.8), the origin of the resulting closed-loop is locally exponentially stable [48]. As a result, if $x(0)$, $\tilde{x}(0)$, and $u(0)$ are sufficiently close to $x^*(0)$, $\tilde{x}^*(0)$, and $u^*(0)$, respectively, by applying the above mentioned controller into system (3.1)-(3.3), the actual trajectory $(x(t), \tilde{x}(t))$ exponentially approaches the reference trajectory $(x^*(t), \tilde{x}^*(t))$. As mentioned in chapter 2, if system (3.10) is uniformly completely controllable, the system is exponentially stabilizable. Thus it is important to check whether or not linearized system (3.10) is uniformly completely controllable. Hence, similarly to the case of nonlinear systems described by ordinary differential equations, the following questions are posed.

**Question 3.1** *What is a class of nonlinear DAS* (3.1)-(3.3) *whose linearizations along trajectories are uniformly completely controllable?*

**Question 3.2** *What is a class of trajectories stated in question 3.1?*

Similarly to the case of nonlinear systems described by ordinary differential equations, we will show that if system (3.1)-(3.3) is algebraically controllable, then every linearized system (3.10) along any (periodic) controllable trajectory is (uniformly) completely controllable in the next section.

On the other hand, the available signal in system (3.1)-(3.3) might be only output signal $y$. In this case, we design a state observer defined by (2.10), where $A(t)$, $B(t)$, and $C(t)$ are defined by (3.11), (3.12), and (3.13), respectively. As mentioned chapter 2, if system (3.10) is uniformly completely controllable and uniformly completely observable, it is expected that we can design a controller and an observer such that system (3.1)-(3.3) becomes locally exponentially stabilizable. Thus it is also important to check whether or not linearized system (3.10) is uniformly completely observable. Hence, similarly to the case of nonlinear systems described by ordinary differential equations, the following questions are posed.

**Question 3.3** *What is a class of nonlinear DAS* (3.1)-(3.3) *whose linearizations along trajectories are uniformly completely observable?*

**Question 3.4** *What is a class of trajectories stated in question 3.3?*

Similarly to the case of nonlinear systems described by ordinary differential equations, we will show that if system (3.1)-(3.3) is algebraically observable, then every linearized system (3.10) along any (periodic) controllable trajectory is (uniformly) completely observable in the next section.

In the above discussion, it is significant that a given reference trajectory composes of a trajectory of system (3.1)-(3.3). Unfortunately, in general, it is difficult to examine whether or not a given reference trajectory composes of a trajectory of system (3.1)-(3.3) because $f$ and $g$ in (3.1) and (3.3) are nonlinear with respect to each variable, we may not be able to obtain a trajectory. As mentioned chapter 2, if a given system is expressed by ordinary differential equations such as (2.1)-(2.2) and the system is differentially flat, we can easily obtain a trajectory (see section 2.1). However, although DAS (3.1)-(3.3) is equivalent to system (3.23) on the set $W$, since one may not be able to get an explicit representation of $\tilde{g}$ as a function of $x$, it might be impossible to examine whether or not system (3.23) is differentially flat in the sense of definition 2.10.

**Example 3.1** Let us consider a simple circuit model shown in Fig. 3.1.
The system is described by

$$L_1 \frac{di_1}{dt} = -e + u, \tag{3.14}$$

$$L_2 \frac{di_2}{dt} = e, \tag{3.15}$$

$$0 = G(e) + i_2 - i_1, \tag{3.16}$$

$$y = i_1, \tag{3.17}$$

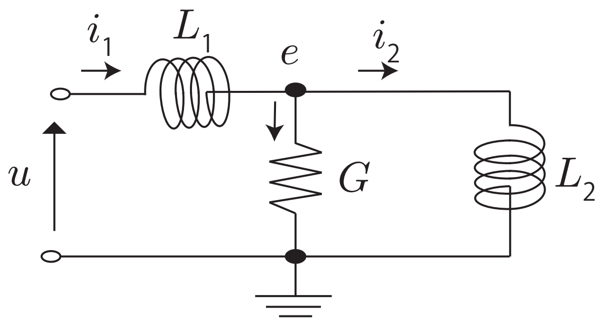



Figure 3.1: A simple circuit

where $i_1$ and $i_2$, $e$, $L_1$ and $L_2$, $G$, $u$, $y$ denote currents, a terminal voltage, inductances of linear inductors, a linear or nonlinear resistance, an input, an output, respectively.

**Case1: Linear Resistance**

Suppose that

$$G(e) = ce,$$

where $c \in \boldsymbol{R}_+$. Then (3.16) implies that

$$e = \frac{1}{c}(i_1 - i_2).$$

Hence system (3.14)-(3.17) is equivalent to

$$\begin{cases} \frac{d}{dt}\begin{pmatrix} i_1 \\ i_2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{L_1 c} & \frac{1}{L_1 c} \\ \frac{1}{L_2 c} & -\frac{1}{L_2 c} \end{pmatrix} \begin{pmatrix} i_1 \\ i_2 \end{pmatrix} + \begin{pmatrix} \frac{1}{L_1} \\ 0 \end{pmatrix} u, \\ y = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} i_1 \\ i_2 \end{pmatrix}. \end{cases} \tag{3.18}$$

From (3.18), by a direct calculation, we have

$$\begin{aligned} i_1 &= L_2 c \frac{di_2}{dt} + i_2, \\ u &= L_2 c \frac{d^2 i_2}{dt^2} + \left(1 + \frac{L_2}{L_1}\right) \frac{di_2}{dt}. \end{aligned}$$

Hence system (3.18) is differentially flat with a flat output $i_2$.

**Case2: PN junction diode**

Suppose that

$$G(e) = I_0(\exp(ke) - 1),$$

where $I_0$, $k \in \boldsymbol{R}_+$ [13]. Then (3.16) implies that

$$e = \frac{1}{k} \log \left( \frac{1}{I_0}(i_1 - i_2) + 1 \right),$$

where $i_1 - i_2 + I_0 > 0$. Hence then (3.14)-(3.17) is equivalent to

$$\begin{cases} \frac{di_1}{dt} = -\frac{1}{L_1 k} \log \left( \frac{1}{I_0}(i_1 - i_2) + 1 \right) + \frac{1}{L_1} u, \\ \frac{di_2}{dt} = \frac{1}{L_2 k} \log \left( \frac{1}{I_0}(i_1 - i_2) + 1 \right), \\ y = i_1. \end{cases} \tag{3.19}$$

From (3.19), by a direct calculation, we have

$$\begin{aligned} i_1 &= I_0 \exp(L_2 k \frac{di_2}{dt}) + i_2 - I_0, \\ u &= L_1 L_2 I_0 k \frac{d^2 i}{dt^2} \exp(L_2 k \frac{di_2}{dt}) + (L_1 + L_2) \frac{di_2}{dt}. \end{aligned}$$

Hence system (3.19) is differentially flat with a flat output $i_2$.

**Case3: Parallel connection of linear resistance and PN junction diode**
Suppose that

$$G(e) = ce + I_0(\exp(ke) - 1).$$

Then by the implicit function theorem, Eq. (3.16) implies that there exists some analytic function $\alpha : \boldsymbol{R}^2 \to \boldsymbol{R}$ such that $e = \alpha(i_1, i_2)$. Then system (3.14)-(3.17) can be transformed into

$$\begin{cases} \frac{di_1}{dt} = -\frac{\alpha(i_1,i_2)}{L_1} + \frac{u}{L_1}, \\ \frac{di_2}{dt} = \frac{\alpha(i_1,i_2)}{L_2}, \\ y = i_1 \end{cases} \tag{3.20}$$

Nevertheless, we cannot obtain an explicit representation of $\alpha$ as a function of $i_1$ and $i_2$. Hence it is impossible to check whether or not system (3.20) is differentially flat in the sense of definition 2.10. ■

Generally speaking, by the implicit function theorem, there exist open sets $X \subset \boldsymbol{R}^n$, $\tilde{X} \subset \boldsymbol{R}^{\tilde{n}}$ satisfying $X \times \tilde{X} \subset W$ and some analytic function $\tilde{g} : X \to \tilde{X}$ such that

$$0 = g(x, \tilde{x}) \Leftrightarrow \tilde{x} = \tilde{g}(x), \tag{3.21}$$

$$\frac{\partial \tilde{g}}{\partial x}(x) = -\left( \frac{\partial g}{\partial \tilde{x}}(x, \tilde{g}(x)) \right)^{-1} \frac{\partial g}{\partial x}(x, \tilde{g}(x)). \tag{3.22}$$

Hence, on the set $W$, system (3.1)-(3.3) is equivalent to

$$\begin{cases} \dot{x} = f(x, \tilde{g}(x), u) =: \tilde{f}(x, u), \\ y = h(x, \tilde{g}(x)) =: \tilde{h}(x). \end{cases} \tag{3.23}$$

However, as mentioned in case 3 of example 3.1, in general, it is impossible to examine whether or not system (3.23) is differentially flat in the sense of definition 2.10 For this reason, we define differential flatness of system (3.1)-(3.3) as follows.

**Definition 3.2** *System (3.1)-(3.3) is called* **differentially flat** *if there exist smooth mappings* $\phi_1 : \boldsymbol{R}^m \times \boldsymbol{R}^m \times \cdots \to \boldsymbol{R}^n$, $\phi_2 : \boldsymbol{R}^m \times \boldsymbol{R}^m \times \cdots \to \boldsymbol{R}^{\tilde{n}}$, $\phi_3 : \boldsymbol{R}^m \times \boldsymbol{R}^m \times \cdots \to \boldsymbol{R}^m$, *and* $\psi : \boldsymbol{R}^n \times \boldsymbol{R}^{\tilde{n}} \times (\boldsymbol{R}^m \times \cdots) \to \boldsymbol{R}^m$ *depending only on a finite number of variables, respectively, such that*

$$v := \psi(x, \tilde{x}, u, \dot{u}, \cdots) \Rightarrow \begin{pmatrix} x \\ \tilde{x} \\ u \end{pmatrix} = \begin{pmatrix} \phi_1(v, \dot{v}, \ddot{v}, \cdots) \\ \phi_2(v, \dot{v}, \ddot{v}, \cdots) \\ \phi_3(v, \dot{v}, \ddot{v}, \cdots) \end{pmatrix}.$$

*In addition, if system (3.1)-(3.3) is differentially flat, the variable $v$ satisfying the above condition is called a* **flat output** *of system (3.1)-(3.3).*

**Remark 3.3** *Let system (3.1)-(3.3) be differentially flat and let a flat output be* $\psi(x, \tilde{x}, u, \dot{u}, \cdots)$. *Then on the set $W$, system* (3.23) *is differentially flat. In particular, then a flat output of system* (3.23) *can be expressed by* $\psi(x, \tilde{g}(x), u, \dot{u}, \cdots)$.
■

**Example 3.2** Let us go back to case 3 in example 3.1. From (3.14)-(3.16), by a direct calculation, we have

$$\begin{cases} i_1 = cL_2\frac{di_2}{dt} + I_0(\exp(kL_2\frac{di_2}{dt}) - 1) + i_2, \\ e = L_2\frac{di_2}{dt}, \\ u = (L_1 + L_2)\frac{di_2}{dt} + L_1L_2(c + I_0k\exp(kL_2\frac{di_2}{dt}))\frac{d^2i_2}{dt^2}. \end{cases} \tag{3.24}$$

Hence system (3.14)-(3.17) is differentially flat with a flat output $i_2$. ■

Assume that system (3.1)-(3.3) is differentially flat with a flat output $v$. Then there exist smooth mappings $\phi_1 : \boldsymbol{R}^m \times \boldsymbol{R}^m \times \cdots \to \boldsymbol{R}^n$, $\phi_2 : \boldsymbol{R}^m \times \boldsymbol{R}^m \times \cdots \to \boldsymbol{R}^{\tilde{n}}$, $\phi_3 : \boldsymbol{R}^m \times \boldsymbol{R}^m \times \cdots \to \boldsymbol{R}^m$ such that

$$\begin{pmatrix} x(t) \\ \tilde{x}(t) \\ u(t) \end{pmatrix} = \begin{pmatrix} \phi_1(v, \dot{v}, \ddot{v}, \cdots) \\ \phi_2(v, \dot{v}, \ddot{v}, \cdots) \\ \phi_3(v, \dot{v}, \ddot{v}, \cdots) \end{pmatrix}.$$

Now assume that $v(t)$ has been defined for all $t \geq 0$. Taking an initial state $x(0) = \phi_1(v(0), \dot{v}(0), \cdots)$ and applying a feedforward control $u(t) = \phi_3(v(t), \dot{v}(t), \cdots)$ for all $t \geq 0$ to system (3.23), by the theorem on uniqueness of solution of ordinary differential equation, we have

$$x(t) = x^*(t) \quad \text{for all} \quad t \geq 0.$$

Since system (3.1)-(3.3) is equivalent to system (3.23) on the set $W$, this means that on the set $W$ under an initial state $(x(0), \tilde{x}(0)) = (\phi_1(v(0), \dot{v}(0), \cdots), \phi_2(v(0), \dot{v}(0), \cdots))$ if we apply the feedforward control $u(t) = \phi_3(v(t), \dot{v}(t), \cdots)$ for all $t \geq 0$ to system (3.1)-(3.3), we have

$$(x(t), \tilde{x}(t)) = (\phi_1(v(t), \dot{v}(t), \cdots), \phi_2(v(t), \dot{v}(t), \cdots)) \quad \text{for all} \quad t \geq 0.$$

Therefore, if we consider a reference trajectory of system (3.1)-(3.3) as

$$(\phi_1(v(t), \dot{v}(t), \cdots), \phi_2(v(t), \dot{v}(t), \cdots)),$$

$$(x(t), \tilde{x}(t), u(t)) = \phi_1(v(t), \dot{v}(t), \cdots), \phi_2(v(t), \dot{v}(t), \cdots), \phi_3(v(t), \dot{v}(t), \cdots))$$

is a trajectory of system (3.1)-(3.3).

## 3.2 Algebraic controllability and algebraic observability of DAS

In order to answer questions 3.1, 3.2, 3.3, and 3.4, we define algebraic controllability (observability) and controllable (observable) trajectory for DAS (3.1)-(3.3). For that, we give some preliminaries. Let $\mathcal{M}_{(x,\tilde{x},u)}$ denote the field of all meromorphic functions depending on a finite number of variables of

$$\left\{ x_i, \tilde{x}_j^{(l)}, u_k^{(l)} \mid 1 \leq i \leq n, 1 \leq j \leq \tilde{n}, 1 \leq k \leq m, l \geq 0 \right\}.$$

The field $\mathcal{M}_{(x,\tilde{x},u)}$ can be endowed with a differential structure determined by system (3.1)-(3.3) as follows:

$$\dot{\phi}(x, \tilde{x}, \dot{\tilde{x}}, \cdots, u, \dot{u}, \cdots) := \frac{\partial \phi}{\partial x} f(x, \tilde{x}, u) + \sum_{l \geq 0} \left( \frac{\partial \phi}{\partial \tilde{x}^{(l)}} \tilde{x}^{(l+1)} + \frac{\partial \phi}{\partial u^{(l)}} u^{(l+1)} \right),$$

where $\phi(x, \tilde{x}, u, \dot{u}, \cdots) \in \mathcal{M}_{(x,\tilde{x},u)}$. We note that on the set $W$, (3.21) and (3.22) imply that $\dot{\tilde{x}} = -\left(\frac{\partial g}{\partial \tilde{x}}\right)^{-1} \frac{\partial g}{\partial x}(x, \tilde{x})\dot{x}$. A vector space $\mathcal{E}_{(x,\tilde{x},u)}$ of differential one forms spanned over $\mathcal{M}_{(x,\tilde{x},u)}$ is defined as

$$\mathcal{E}_{(x,\tilde{x},u)} := \text{span}_{\mathcal{M}_{(x,\tilde{x},u)}} \left\{ dx_i, d\tilde{x}_j^{(l)}, du_k^{(l)} \mid 1 \leq i \leq n, 1 \leq j \leq \tilde{n}, 1 \leq k \leq m, l \geq 0 \right\}.$$

Then for any $\phi \in \mathcal{M}_{(x,\tilde{x},u)}$, differential $d : \mathcal{M}_{(x,\tilde{x},u)} \to \mathcal{E}_{(x,\tilde{x},u)}$ is defined as

$$d\phi := \frac{\partial \phi}{\partial x}dx + \sum_{l\geq 0}\left(\frac{\partial \phi}{\partial \tilde{x}^{(l)}}d\tilde{x}^{(l)} + \frac{\partial \phi}{\partial u^{(l)}}du^{(l)}\right). \tag{3.25}$$

Let $\mathcal{D}_{(x,\tilde{x},u)} := \mathcal{M}_{(x,\tilde{x},u)}\left[\frac{d}{dt}\right]$. If we take $\alpha = \sum_{i=0}^{m}\alpha_i \frac{d^i}{dt^i} \in \mathcal{D}_{(x,\tilde{x},u)}$, where $\alpha_i \in \mathcal{M}_{(x,\tilde{x},u)}$, then $\frac{d}{dt}\alpha$ is defined as

$$\frac{d}{dt}\alpha := \sum_{i=0}^{m}\left(\alpha_i \frac{d^{i+1}}{dt^{i+1}} + \dot{\alpha}_i \frac{d^i}{dt^i}\right) \in \mathcal{D}_{(x,\tilde{x},u)}.$$

Thus $\mathcal{D}_{(x,\tilde{x},u)}$ is a left skew polynomial ring, and thus elements of $\mathcal{D}_{(x,\tilde{x},u)}$ can act on the vector space $\mathcal{E}_{(x,\tilde{x},u)}$ (see appendix C). In fact, the vector space $\mathcal{E}_{(x,\tilde{x},u)}$ can be endowed with a differential structure by defining a derivative operator $\frac{d}{dt}$ as follows:

$$\begin{aligned}&\frac{d}{dt}\left(\sum_{i=1}^{n} a_i dx_i + \sum_{l\geq 0}(\sum_{j=1}^{\tilde{n}} b_{j,l}d\tilde{x}_j^{(l)} + \sum_{k=1}^{m} c_{k,l}du_k^{(l)})\right)\\ &:= \sum_{i=1}^{n}(\dot{a}_i dx_i + a_i d\dot{x}_i) + \sum_{l\geq 0}\left(\sum_{j=1}^{\tilde{n}}(\dot{b}_{j,l}d\tilde{x}_j^{(l)} + b_{j,l}d\dot{\tilde{x}}_j^{(l+1)}) + \sum_{k=1}^{m}(\dot{c}_{k,l}du_k^{(l)} + c_{k,l}du_k^{(l+1)})\right).\end{aligned}$$

where $\sum_{i=1}^{n} a_i dx_i + \sum_{l\geq 0}(\sum_{j=1}^{\tilde{n}} b_{j,l}d\tilde{x}_j^{(l)} + \sum_{k=1}^{m} c_{k,l}du_k^{(l)}) \in \mathcal{E}_{(x,\tilde{x},u)}$. Furthermore, $\mathcal{D}_{(x,\tilde{x},u)}$ is simple and a non-commutative Euclidean domain (see proposition B.3 in appendix B).

Now, differentiating both sides of system (3.1)-(3.2), we have

$$P^c_{(x,\tilde{x},u)}\begin{pmatrix} dx \\ d\tilde{x} \\ du \end{pmatrix} = 0,$$

where

$$P^c_{(x,\tilde{x},u)} := \begin{pmatrix} \frac{d}{dt}I - \frac{\partial f}{\partial x}(x,\tilde{x},u) & -\frac{\partial f}{\partial \tilde{x}}(x,\tilde{x},u) & -\frac{\partial f}{\partial u}(x,\tilde{x},u) \\ \frac{\partial g}{\partial x}(x,\tilde{x}) & \frac{\partial g}{\partial \tilde{x}}(x,\tilde{x}) & 0 \end{pmatrix}. \tag{3.26}$$

Since $f$ and $g$ are meromorphic with respect to each variable, coefficients of polynomials of each element of $P^c_{(x,\tilde{x},u)}$ are meromorphic functions. Thus $P^c_{(x,\tilde{x},u)} \in \mathcal{D}_{(x,\tilde{x},u)}^{(n+\tilde{n})\times(n+\tilde{n}+m)}$.

**Definition 3.3** *System (3.1)-(3.3) is called* **algebraically controllable** *if $P^c_{(x,\tilde{x},u)}$ defined by* (3.26) *is hyper-regular, that is, there exist unimodular matrices $U_{(x,\tilde{x},u)} \in \mathcal{D}_{(x,\tilde{x},u)}^{(n+\tilde{n})\times(n+\tilde{n})}$ and $V_{(x,\tilde{x},u)} \in \mathcal{D}_{(x,\tilde{x},u)}^{(n+\tilde{n}+m)\times(n+\tilde{n}+m)}$ such that*

$$U_{(x,\tilde{x},u)}P^c_{(x,\tilde{x},u)}V_{(x,\tilde{x},u)} = \begin{pmatrix} I_{n+\tilde{n}} & 0_{(n+\tilde{n})\times m}\end{pmatrix}. \tag{3.27}$$

**Remark 3.4** *Similarly to the case of nonlinear systems* (2.1)-(2.2) *described by ordinary differential equations, algebraic controllability for DAS* (3.1)-(3.3) *is invariant under an analytic coordinate transformation.* ■

Similarly to the case of (2.1)-(2.2), we define controllable trajectory of system (3.1)-(3.3). To define controllable trajectory, we need to prepare some definitions.

**Definition 3.4** *Let $R_{(x,\tilde{x},u)} \in \mathcal{D}^{a\times b}_{(x,\tilde{x},u)}$ and let $(x^*(t), \tilde{x}^*(t), u^*(t)) \in \boldsymbol{R}^n \times \boldsymbol{R}^{\tilde{n}} \times \boldsymbol{R}^m$ be a trajectory of system* (3.1)-(3.3)*. The matrix $R_{(x^*(t),\tilde{x}^*(t),u^*(t))}$ is defined by substituting $x^*(t)$, $\tilde{x}^*(t)$, and $u^*(t)$ into $x$, $\tilde{x}$, and $u$ in $R_{(x,\tilde{x},u)}$, respectively.*

**Definition 3.5** *Let $R_{(x,\tilde{x},u)} \in \mathcal{D}^{a\times b}_{(x,\tilde{x},u)}$ and let $(x^*(t), \tilde{x}^*(t), u^*(t)) \in \boldsymbol{R}^n \times \boldsymbol{R}^{\tilde{n}} \times \boldsymbol{R}^m$ be a trajectory of system* (3.1)-(3.3)*. The matrix $R_{(x^*(t),\tilde{x}^*(t),u^*(t))}$ is called* **bounded** *if every coefficient function of each polynomial element of $R_{(x^*(t),\tilde{x}^*(t),u^*(t))}$ is bounded on $\boldsymbol{R}$.*

For an algebraically controllable system (3.1)-(3.3), controllable trajectory is composed of functions in $C^\infty_{\mathrm{pw}}$ and the state trajectory part is contained in the set $W$ for all $t \in \boldsymbol{R}$.

**Definition 3.6** *Suppose that system (3.1)-(3.3) is algebraically controllable. Then a trajectory $(x^*(t), \tilde{x}^*(t), u^*(t))$ of system (3.1)-(3.3) is called a* **controllable trajectory** *if the following conditions are satisfied:*

1. *$(x^*, \tilde{x}^*, u^*) \in (C^\infty_{\mathrm{pw}})^n \times (C^\infty_{\mathrm{pw}})^{\tilde{n}} \times (C^\infty_{\mathrm{pw}})^m$ and $(x^*(t), \tilde{x}^*(t)) \in W$ on $\boldsymbol{R}$.*
2. *The matrix $P^c_{(x^*(t),\tilde{x}^*(t),u^*(t))}$ is bounded.*
3. *There exist unimodular matrices $U_{(x,\tilde{x},u)} \in \mathcal{D}^{(n+\tilde{n})\times(n+\tilde{n})}_{(x,\tilde{x},u)}$ and $V_{(x,\tilde{x},u)} \in \mathcal{D}^{(n+\tilde{n}+m)\times(n+\tilde{n}+m)}_{(x,\tilde{x},u)}$ satisfying* (3.27) *such that*

$$U_{(x^*(t),\tilde{x}^*(t),u^*(t))}, U^{-1}_{(x^*(t),\tilde{x}^*(t),u^*(t))} \in \left(C^\infty_{\mathrm{a.e.}}\left[\frac{d}{dt}\right]\right)^{(n+\tilde{n})\times(n+\tilde{n})},$$
$$V_{(x^*(t),\tilde{x}^*(t),u^*(t))}, V^{-1}_{(x^*(t),\tilde{x}^*(t),u^*(t))} \in \left(C^\infty_{\mathrm{a.e.}}\left[\frac{d}{dt}\right]\right)^{(n+\tilde{n}+m)\times(n+\tilde{n}+m)}.$$

**Example 3.3** Let us go back to example 3.1 and consider the case 3. Then system (3.14)-(3.17) is equivalent to

$$\frac{di_1}{dt} = -\frac{e}{L_1} + \frac{u}{L_1}, \tag{3.28}$$
$$\frac{di_2}{dt} = \frac{e}{L_2}, \tag{3.29}$$
$$0 = ce + I_0(\exp(ke) - 1) + i_2 - i_1, \tag{3.30}$$
$$y = i_1. \tag{3.31}$$

Differentiating both sides (3.28)-(3.30), we have

$$\underbrace{\begin{pmatrix} \frac{d}{dt} & 0 & \frac{1}{L_1} & -\frac{1}{L_1} \\ 0 & \frac{d}{dt} & -\frac{1}{L_2} & 0 \\ -1 & 1 & c + I_0 k \exp(ke) & 0 \end{pmatrix}}_{P^c_{(i_1,i_2,e,u)}} \begin{pmatrix} di_1 \\ di_2 \\ de \\ du \end{pmatrix} = 0.$$

Repeating elementary column operations for $P^c_{(i_1,i_2,e,u)}$, we have

$$P^c_{(i_1,i_2,e,u)} V_{(i_1,i_2,e,u)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix},$$

where

$$V_{(i_1,i_2,e,u)} := \begin{pmatrix} 0 & -L_2\alpha & -1 & 1 + L_2\alpha\frac{d}{dt} \\ 0 & 0 & 0 & 1 \\ 0 & -L_2 & 0 & L_2\frac{d}{dt} \\ -L_1 & -L_2(L_1\frac{d}{dt}\alpha + 1) & -L_1\frac{d}{dt} & L_1\frac{d}{dt} + (L_1\frac{d}{dt}\alpha + 1)L_2\frac{d}{dt} \end{pmatrix},$$

and where

$$\alpha := c + I_0 k \exp(ke).$$

Hence system (3.31) is algebraically controllable.

Furthermore,

$$V^{-1}_{(i_1,i_2,e,u)} = \begin{pmatrix} \frac{d}{dt} & 0 & \frac{1}{L_1} & -\frac{1}{L_1} \\ 0 & \frac{d}{dt} & -\frac{1}{L_2} & 0 \\ -1 & 1 & \alpha & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Viewing each element of $P^c_{(i_1,i_2,e,u)}$, $V_{(i_1,i_2,e,u)}$, and $V^{-1}_{(i_1,i_2,e,u)}$, we can know that any smooth trajectory $(i_1^*(t), i_2^*(t), e^*(t), u^*(t))$ of system (3.14)-(3.17) such that $\exp(ke^*(t))$ is bounded on $\boldsymbol{R}$ is a controllable trajectory. ■

Let $(x^*(t), \tilde{x}^*(t), u^*(t))$ be any trajectory of system (3.1)-(3.3) such that $P^c_{(x^*(t),\tilde{x}^*(t),u^*(t))}$ is bounded. Then we can define the behavior

$$\mathcal{B}_{(x^*(t),\tilde{x}^*(t),u^*(t))} := \left\{ (x_\epsilon, \tilde{x}_\epsilon, u_\epsilon) \in (C^\infty_{\text{a.e.}})^n \times (C^\infty_{\text{a.e.}})^{\tilde{n}} \times (C^\infty_{\text{a.e.}})^m \;\middle|\; P^c_{(x^*(t),\tilde{x}^*(t)u^*(t))} \begin{pmatrix} x_\epsilon \\ \tilde{x}_\epsilon \\ u_\epsilon \end{pmatrix} = 0 \right\}. \tag{3.32}$$

Similarly to lemma 2.16, we have the following lemma.

**Lemma 3.7** *Suppose that system* (3.1)-(3.3) *is algebraically controllable. Let* $(x^*(t), \tilde{x}^*(t), u^*(t))$ *be any controllable trajectory. Then behavior* $\mathcal{B}_{(x^*(t),\tilde{x}^*(t),u^*(t))}$ *is controllable.*

Now, we can relate algebraic controllability and complete controllability in the same way as theorem 2.6.

**Theorem 3.1** *Suppose that system (3.1)-(3.3) is algebraically controllable. Then every linearized system* (3.10) *along any controllable trajectory* $(x^*(t), \tilde{x}^*(t), u^*(t))$ *of system (3.1)-(3.3) is completely controllable.*

**Proof** By lemma 3.7, the behavior $\mathcal{B}_{(x^*(t),\tilde{x}^*(t),u^*(t))}$ defined by (3.32) is controllable. Since $P^c_{(x^*(t),\tilde{x}^*(t),u^*(t))}\begin{pmatrix} x_\epsilon \\ \tilde{x}_\epsilon \\ u_\epsilon \end{pmatrix} = 0$ is equivalent to

$$\begin{cases} \dot{x}_\epsilon = \frac{\partial f}{\partial x}(x^*(t), \tilde{x}^*_\epsilon(t), u^*(t))x_\epsilon + \frac{\partial f}{\partial \tilde{x}}(x^*(t), \tilde{x}^*_\epsilon(t), u^*(t))\tilde{x}_\epsilon \\ \qquad + \frac{\partial f}{\partial u}(x^*(t), \tilde{x}^*_\epsilon(t), u^*(t))u_\epsilon, \\ \frac{\partial g}{\partial x}(x^*(t), \tilde{x}^*(t))x_\epsilon + \frac{\partial g}{\partial \tilde{x}}(x^*(t), \tilde{x}^*(t))\tilde{x}_\epsilon = 0. \end{cases} \tag{3.33}$$

Moreover, since $(x^*(t), \tilde{x}^*(t), u^*(t))$ is a controllable trajectory of system (3.1)-(3.3), $(x^*(t), \tilde{x}^*(t)) \in W$ on $\boldsymbol{R}$. Thus $\frac{\partial g}{\partial \tilde{x}}(x^*(t), \tilde{x}^*(t))$ is invertible for all $t \in \boldsymbol{R}$. Hence (3.33) is equivalent to linear system (3.10). Therefore similarly to the proof of theorem 2.6, we have the conclusion. □

If $P^c_{(x^*(t),u^*(t))}$ is bounded and $\frac{1}{\det \frac{\partial g}{\partial \tilde{x}}(x^*(t),u^*(t))}$ is bounded on $\boldsymbol{R}$, $A(\cdot)$ and $B(\cdot)$ defined as (3.11) and (3.12), respectively, are bounded on $\boldsymbol{R}$. Hence in the same way as corollary 2.7, we have the following corollary.

**Corollary 3.2** *Suppose that system (3.1)-(3.3) is algebraically controllable. Then every linearized system* (3.10) *along any periodic controllable trajectory* $(x^*(t), \tilde{x}^*(t), u^*(t))$ *of system (3.1)-(3.3) such that* $\frac{1}{\det \frac{\partial g}{\partial \tilde{x}}(x^*(t),u^*(t))}$ *is bounded on* $\boldsymbol{R}$ *is uniformly completely controllable.*

Next, we introduce a concept of algebraic observability of system (3.1)-(3.3). Differentiating both sides of system (3.1)-(3.3), we have

$$P^o_{(x,\tilde{x},u)}\begin{pmatrix} dx \\ d\tilde{x} \end{pmatrix} = Q_{(x,\tilde{x},u)}\begin{pmatrix} du \\ dy \end{pmatrix}, \tag{3.34}$$

where

$$P^o_{(x,\tilde{x},u)} := \begin{pmatrix} \frac{d}{dt}I - \frac{\partial f}{\partial x}(x,\tilde{x},u) & -\frac{\partial f}{\partial \tilde{x}}(x,\tilde{x},u) \\ \frac{\partial g}{\partial x}(x,\tilde{x}) & \frac{\partial g}{\partial \tilde{x}}(x,\tilde{x}) \\ -\frac{\partial h}{\partial x}(x,\tilde{x}) & -\frac{\partial h}{\partial \tilde{x}}(x,\tilde{x}) \end{pmatrix} \in \mathcal{D}^{(n+\tilde{n}+p)\times(n+\tilde{n})}_{(x,\tilde{x},u)}, \tag{3.35}$$

$$Q_{(x,\tilde{x},u)} := \begin{pmatrix} \frac{\partial f}{\partial u}(x,\tilde{x},u) & 0 \\ 0 & 0 \\ 0 & -I \end{pmatrix} \in \mathcal{D}^{(n+\tilde{n}+p)\times(m+p)}_{(x,\tilde{x},u)}.$$

In the same way as algebraic controllability, algebraic observability is defined.

**Definition 3.8** *System* (3.1)-(3.3) *is called* **algebraically observable** *if* $P_{(x,\tilde{x},u)}$ *defined by* (3.35) *is hyper-regular, that is, there exist unimodular matrices* $U_{(x,\tilde{x},u)} \in \mathcal{D}_{(x,\tilde{x},u)}^{(n+\tilde{n}+p)\times(n+\tilde{n}+p)}$ *and* $V_{(x,\tilde{x},u)} \in \mathcal{D}_{(x,\tilde{x},u)}^{(n+\tilde{n})\times(n+\tilde{n})}$ *such that*

$$U_{(x,\tilde{x},u)} P^o_{(x,\tilde{x},u)} V_{(x,\tilde{x},u)} = \begin{pmatrix} I_{n+\tilde{n}} \\ 0 \end{pmatrix}. \tag{3.36}$$

**Remark 3.5** *Similarly to the case of nonlinear systems* (2.1)-(2.2) *described by ordinary differential equations, algebraic observability for DAS* (3.1)-(3.3) *is invariant under an analytic coordinate transformation.* ■

Similarly to the case of algebraic controllability, in order to relate algebraic observability and uniform complete observability, we define observable trajectory of system (3.1)-(3.3).

**Definition 3.9** *Suppose that system* (3.1)-(3.3) *is algebraically observable. Then a trajectory* $(x^*(t), \tilde{x}^*(t), u^*(t))$ *of system* (3.1)-(3.3) *is called an* **observable trajectory** *if the following conditions are satisfied:*

1. $(x^*, \tilde{x}^*, u^*) \in (C^\infty_{\mathrm{pw}})^n \times (C^\infty_{\mathrm{pw}})^{\tilde{n}} \times (C^\infty_{\mathrm{pw}})^m$ *and* $(x^*(t), \tilde{x}^*(t)) \in W$ *on* $\boldsymbol{R}$.
2. *The matrices* $P^o_{(x^*(t),\tilde{x}^*(t),u^*(t))}$ *and* $Q^o_{(x^*(t),\tilde{x}^*(t),u^*(t))}$ *are bounded.*
3. *There exist unimodular matrices* $U_{(x,\tilde{x},u)} \in \mathcal{D}_{(x,\tilde{x},u)}^{(n+\tilde{n}+p)\times(n+\tilde{n}+p)}$ *and* $V_{(x,\tilde{x},u)} \in \mathcal{D}_{(x,\tilde{x},u)}^{(n+\tilde{n})\times(n+\tilde{n})}$ *satisfying* (2.36) *such that*

$$U_{(x^*(t),\tilde{x}^*(t),u^*(t))}, U^{-1}_{(x^*(t),\tilde{x}^*(t),u^*(t))} \in \left(C^\infty_{\mathrm{a.e.}}\left[\frac{d}{dt}\right]\right)^{(n+\tilde{n}+p)\times(n+\tilde{n}+p)},$$

$$V_{(x^*(t),\tilde{x}^*(t),u^*(t))}, V^{-1}_{(x^*(t),\tilde{x}^*(t),u^*(t))} \in \left(C^\infty_{\mathrm{a.e.}}\left[\frac{d}{dt}\right]\right)^{(n+\tilde{n})\times(n+\tilde{n})}.$$

As dualities of theorem 3.1 and corollary 3.2, we have the following theorem and corollary, respectively.

**Theorem 3.3** *Suppose that system* (3.1)-(3.3) *is algebraically observable. Then every linearized system* (3.10) *along any observable trajectory* $(x^*(t), \tilde{x}^*(t), u^*(t))$ *of system* (3.1)-(3.3) *is completely observable.*

**Proof** Since system (3.1)-(3.3) is algebraically observable, there exist unimodular matrices $U_{(x,\tilde{x},u)} \in \mathcal{D}_{(x,\tilde{x},u)}^{(n+\tilde{n}+p)\times(n+\tilde{n}+p)}$ and $V_{(x,\tilde{x},u)} \in \mathcal{D}_{(x,\tilde{x},u)}^{(n+\tilde{n})\times(n+\tilde{n})}$ satisfying (3.36). Hence we have

$$V^T_{(x,\tilde{x},u)} (P^o_{(x,\tilde{x},u)})^T U^T_{(x,\tilde{x},u)} = \begin{pmatrix} I_{n+\tilde{n}} & 0 \end{pmatrix}.$$

Let $(x^*(t), \tilde{x}^*(t), u^*(t))$ be any observable trajectory of system (3.1)-(3.3). Now consider

$$(P^o_{(x^*(\bar{t}),\tilde{x}^*(\bar{t}),u^*(\bar{t}))})^T \begin{pmatrix} \bar{x}_\epsilon \\ \bar{\tilde{x}}_\epsilon \\ \bar{u}_\epsilon \end{pmatrix} = 0, \tag{3.37}$$

where $\bar{x}_\epsilon \in (C^\infty_{\text{a.e.}})^n$, $\bar{\tilde{x}}_\epsilon \in (C^\infty_{\text{a.e.}})^{\tilde{n}}$, $\bar{u}_\epsilon \in (C^\infty_{\text{a.e.}})^p$, and $\bar{t} := -t$. Eq. (3.37) is equivalent to

$$\begin{cases} \dot{\bar{x}}_\epsilon = \left(\frac{\partial f}{\partial x}(x^*(\bar{t}), \tilde{x}^*(\bar{t}), u^*(\bar{t}))\right)^T \bar{x}_\epsilon - \left(\frac{\partial g}{\partial x}(x^*(\bar{t}), \tilde{x}^*(\bar{t}))\right)^T \bar{\tilde{x}}_\epsilon \\ \qquad + \left(\frac{\partial h}{\partial x}(x^*(\bar{t}), \tilde{x}^*(\bar{t}))\right)^T \bar{u}_\epsilon, \\ \left(\frac{\partial f}{\partial \tilde{x}}(x^*(\bar{t}), \tilde{x}^*(\bar{t}), u^*(\bar{t}))\right)^T \bar{x}_\epsilon - \left(\frac{\partial g}{\partial \tilde{x}}(x^*(\bar{t}), \tilde{x}^*(\bar{t})\right)^T \bar{\tilde{x}}_\epsilon + \left(\frac{\partial h}{\partial \tilde{x}}(x^*(\bar{t}), \tilde{x}^*(\bar{t}))\right)^T \bar{u}_\epsilon = 0. \end{cases} \tag{3.38}$$

Since $(x^*(t), \tilde{x}^*(t), u^*(t))$ is an observable trajectory of system (3.1)-(3.3), $(x^*(\bar{t}), \tilde{x}^*(\bar{t})) \in W$ on $\boldsymbol{R}$. Thus, $\frac{\partial g}{\partial \tilde{x}}(x^*(\bar{t}), \tilde{x}^*(\bar{t}))$ is invertible for all $\bar{t} \in \boldsymbol{R}$. Hence (3.38) is equivalent to

$$\frac{d\bar{x}_\epsilon}{d\bar{t}} = A(\bar{t})^T \bar{x}_\epsilon + C(\bar{t})^T \bar{u}_\epsilon,$$

where $A(\cdot)$ and $C(\cdot)$ are defined in (3.10). Therefore similarly to the proof of theorem 2.11, we have the conclusion. □

**Corollary 3.4** *Suppose that system* (3.1)-(3.3) *is algebraically observable. Then every linearized system* (3.10) *along any periodic observable trajectory* $(x^*(t), \tilde{x}^*(t), u^*(t))$ *of system* (3.1)-(3.3) *such that* $\frac{1}{\det \frac{\partial g}{\partial \tilde{x}}(x^*(t),\tilde{x}^*(t))}$ *is bounded on* $\boldsymbol{R}$ *is uniformly completely observable.*

**Example 3.4** Let us go back to example 3.3. Differentiating both sides (3.28)-(3.31), we have

$$\underbrace{\begin{pmatrix} \frac{d}{dt} & 0 & \frac{1}{L_1} \\ 0 & \frac{d}{dt} & -\frac{1}{L_2} \\ -1 & 1 & c + I_0 k \exp(ke) \\ -1 & 0 & 0 \end{pmatrix}}_{P^o_{(i_1,i_2,e,u)}} \begin{pmatrix} di_1 \\ di_2 \\ de \end{pmatrix} = \begin{pmatrix} \frac{1}{L_1} & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} du \\ dy \end{pmatrix}$$

Repeating elementary row operations for $P^o_{(i_1,i_2,e,u)}$, we have

$$U_{(i_1,i_2,e,u)} P^o_{(i_1,i_2,e,u)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

where

$$U_{(i_1,i_2,e,u)} := \begin{pmatrix} 0 & 0 & 0 & -1 \\ -\alpha L_1 & 0 & 1 & -1-\alpha L_1 \frac{d}{dt} \\ L_1 & 0 & 0 & L_1 \frac{d}{dt} \\ \frac{L_1}{L_2} + L_1 \frac{d}{dt}\alpha & 1 & -\frac{d}{dt} & \left(\frac{1}{L_2} + \frac{d}{dt}\alpha\right) L_1 \frac{d}{dt} + \frac{d}{dt} \end{pmatrix},$$

and where

$$\alpha := c + I_0 k \exp(ke).$$

Hence system (3.28)-(3.31) is algebraically observable. Furthermore, we have

$$U^{-1}_{(i_1,i_2,e,u)} = \begin{pmatrix} \frac{d}{dt} & 0 & \frac{1}{L_1} & 0 \\ 0 & \frac{d}{dt} & -\frac{1}{L_2} & 1 \\ -1 & 1 & \alpha & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

Viewing each element of $P^o_{(i_1,i_2,e,u)}$, $U_{(i_1,i_2,e,u)}$, and $U^{-1}_{(i_1,i_2,e,u)}$, we can know that any smooth trajectory $(i_1^*(t), i_2^*(t), e^*(t), u^*(t))$ of system (3.14)-(3.17) such that $\exp(ke^*(t))$ is bounded on $\boldsymbol{R}$ is an observable trajectory.

By the discussion in example 3.2, system (3.28)-(3.31) is differentially flat with a flat output $i_2$. Hence we can easily find a periodic controllable and observable trajectory. In fact, as an example of such a trajectory, from relation (3.24) we have

$$\begin{cases} i_1(t) = cL_2 \cos t + I_0 \left(\exp(kL_2 \cos t) - 1\right) + \sin t, \\ i_2(t) = \sin t, \\ e(t) = L_2 \cos t, \\ u(t) = (L_1 + L_2) \cos t \\ \qquad\quad -L_1 L_2 (c + I_0 k \exp(kL_2 \cos t)) \sin t. \end{cases} \tag{3.39}$$

■

## 3.3 Differential flatness of DAS

In this section, we consider differential flatness of a nonlinear differential algebraic system

$$F(w, \dot{w}, \cdots) = 0, \tag{3.40}$$

where $w = (w_1, \cdots, w_q)$ and $F : \boldsymbol{R}^q \times \boldsymbol{R}^q \times \cdots \to \boldsymbol{R}^l$ is a smooth mapping depending only on a finite number of variables of $\{w, \dot{w}, \ddot{w}, \cdots\}$. We note that system (3.40) does not distinguish state, input, and output variables. As an extension of differential flatness of [22,23], we define differential flatness of system (3.40).

**Definition 3.10** *System* (3.40) *is called* **differentially flat** *if there exist smooth mappings* $\psi : \boldsymbol{R}^q \times \boldsymbol{R}^q \times \cdots \to \boldsymbol{R}^p$ *and* $\phi : \boldsymbol{R}^p \times \boldsymbol{R}^p \times \cdots \to \boldsymbol{R}^q$ *depending only on a finite number of variables of* $\{w, \dot{w}, \cdots\}$ *and* $\{y, \dot{y}, \cdots\}$*, respectively, such that*

1.
$$y := \psi(w, \dot{w}, \cdots) \Longrightarrow w = \phi(y, \dot{y}, \cdots),$$

2. $y_1, \cdots, y_p$ *are differentially independent, that is, for an arbitrary positive integer* $n$*,*
$$c_1^0 dy_1 + \cdots + c_p^0 dy_p + \cdots + c_1^n dy_1^{(n)} + \cdots + c_p^n dy_p^{(n)} = 0$$
*implies that*
$$c_1^0 = \cdots = c_p^0 = \cdots = c_1^n = \cdots = c_p^n = 0,$$
*where* $c_i^j$ *is a smooth function.*

*In addition, if system* (3.40) *is differentially flat, the variable* $y$ *satisfying the above conditions is called a* **flat output***.*

**Remark 3.6** *In the case of systems* (2.1)-(2.2) *and* (3.1)-(3.3)*, condition 2 is not required if input variables are differentially independent. That is, we implicitly assume differential independence of input variables for systems* (2.1)-(2.2) *and* (3.1)-(3.3)*.* ■

We note that a flat output of differentially flat system (3.40) is not unique. If we could find a flat output of differentially flat system (3.40), we can obtain other flat outputs. In fact, we have the following theorem.

**Theorem 3.5** *Suppose that system* (3.40) *is differentially flat with a flat output* $y \in \boldsymbol{R}^p$*. Then*

$$\tilde{y} := \alpha(y)$$

*is also a flat output of system* (3.40)*, where* $\alpha : \boldsymbol{R}^p \to \boldsymbol{R}^p$ *is an arbitrary smooth mapping such that* $\frac{\partial \alpha}{\partial y}$ *is invertible at every point* $y \in \boldsymbol{R}^p$*.*

**Proof** First, we show that there exist smooth mappings $\tilde{\psi} : \boldsymbol{R}^q \times \boldsymbol{R}^q \times \cdots \to \boldsymbol{R}^p$ and $\tilde{\phi} : \boldsymbol{R}^p \times \boldsymbol{R}^p \times \cdots \to \boldsymbol{R}^q$ such that

$$\tilde{y} = \tilde{\psi}(w, \dot{w}, \cdots) \Rightarrow w = \tilde{\phi}(\tilde{y}, \dot{\tilde{y}}, \cdots). \tag{3.41}$$

Since $\alpha : \boldsymbol{R}^p \to \boldsymbol{R}^p$ is smooth such that $\frac{\partial \alpha}{\partial y}$ is invertible at every point $y \in \boldsymbol{R}^p$, by the inverse function theorem there exists a smooth mapping $\tilde{\alpha} : \boldsymbol{R}^p \to \boldsymbol{R}^p$ such that

$$y = \tilde{\alpha}(\tilde{y}).$$

Since system (3.40) is differentially flat, there exist smooth mappings $\psi : \boldsymbol{R}^q \times \boldsymbol{R}^q \times \cdots \to \boldsymbol{R}^p$ and $\phi : \boldsymbol{R}^p \times \boldsymbol{R}^p \times \cdots \to \boldsymbol{R}^q$ such that $y = \psi(w, \dot{w}, \cdots) \Rightarrow w = \phi(y, \dot{y}, \cdots)$. Hence putting

$$\tilde{\psi} := \alpha \circ \psi \quad \text{and} \quad \tilde{\phi}(\tilde{y}, \dot{\tilde{y}}, \cdots) := \phi(\tilde{\alpha}(\tilde{y}), \frac{\partial \tilde{\alpha}}{\partial \tilde{y}} \dot{\tilde{y}}, \cdots),$$

we have (3.41).

Next, we show that $\tilde{y}_1, \cdots, \tilde{y}_p$ are differentially independent. Now suppose that for an arbitrary positive integer $n$,

$$c_1^0 d\tilde{y}_1 + \cdots + c_p^0 d\tilde{y}_p + \cdots + c_1^n d\tilde{y}_1^{(n)} + \cdots + c_p^n d\tilde{y}_p^{(n)} = 0, \tag{3.42}$$

where $c_i^j$ is a smooth function. Since $d\tilde{y}_i^{(k)} = \sum_{j=1}^p \frac{\partial \alpha_i}{\partial y_j} dy_j^{(k)} + \sum_{j,l=1}^n \frac{\partial^2 \alpha_i}{\partial y_j \partial y_l} \dot{y}_l dy_j^{(k-1)} + \cdots$, $1 \le i \le p$, $1 \le k \le n$, (3.42) is equivalent to the form

$$\begin{aligned}
&\left( c_1^0 \frac{\partial \alpha_1}{\partial y_1} + \cdots + c_p^0 \frac{\partial \alpha_p}{\partial y_1} + c_1^1 \sum_{k=1}^p \frac{\partial^2 \alpha_1}{\partial y_1 \partial y_k} \dot{y}_k + \cdots + c_p^1 \sum_{k=1}^p \frac{\partial^2 \alpha_p}{\partial y_1 \partial y_k} \dot{y}_k + \cdots \right) dy_1 + \cdots \\
+ &\left( c_1^0 \frac{\partial \alpha_1}{\partial y_p} + \cdots + c_p^0 \frac{\partial \alpha_p}{\partial y_p} + c_1^1 \sum_{k=1}^p \frac{\partial^2 \alpha_1}{\partial y_p \partial y_k} \dot{y}_k + \cdots + c_p^1 \sum_{k=1}^p \frac{\partial^2 \alpha_p}{\partial y_p \partial y_k} \dot{y}_k + \cdots \right) dy_p + \cdots \\
+ &\left( c_1^{n-1} \frac{\partial \alpha_1}{\partial y_1} + \cdots + c_p^{n-1} \frac{\partial \alpha_p}{\partial y_1} + c_1^n \sum_{k=1}^p \frac{\partial^2 \alpha_1}{\partial y_1 \partial y_k} \dot{y}_k + \cdots + c_p^n \sum_{k=1}^p \frac{\partial^2 \alpha_p}{\partial y_1 \partial y_k} \dot{y}_k \right) dy_1^{(n-1)} + \cdots \\
+ &\left( c_1^{n-1} \frac{\partial \alpha_1}{\partial y_p} + \cdots + c_p^{n-1} \frac{\partial \alpha_p}{\partial y_p} + c_1^n \sum_{k=1}^p \frac{\partial^2 \alpha_1}{\partial y_p \partial y_k} \dot{y}_k + \cdots + c_p^n \sum_{k=1}^p \frac{\partial^2 \alpha_p}{\partial y_p \partial y_k} \dot{y}_k \right) dy_p^{(n-1)} + \cdots \\
+ &\left( c_1^n \frac{\partial \alpha_1}{\partial y_1} + \cdots + c_p^n \frac{\partial \alpha_p}{\partial y_1} \right) dy_1^{(n)} + \cdots + \left( c_1^n \frac{\partial \alpha_1}{\partial y_p} + \cdots + c_p^n \frac{\partial \alpha_p}{\partial y_p} \right) dy_p^{(n)} = 0.
\end{aligned}$$

Since $y$ is a flat output of system (3.40), $y_1 \cdots, y_p$ are differentially independent. Thus the coefficients of $dy_1^{(n)}, \cdots, dy_p^{(n)}$ are equal to zero, that is,

$$\begin{cases} c_1^n \frac{\partial \alpha_1}{\partial y_1} + \cdots + c_p^n \frac{\partial \alpha_p}{\partial y_1} = 0 \\ \qquad \vdots \\ c_1^n \frac{\partial \alpha_1}{\partial y_p} + \cdots + c_p^n \frac{\partial \alpha_p}{\partial y_p} = 0 \end{cases} \Leftrightarrow \begin{pmatrix} \frac{\partial \alpha_1}{\partial y_1} & \cdots & \frac{\partial \alpha_p}{\partial y_1} \\ \vdots & & \vdots \\ \frac{\partial \alpha_1}{\partial y_p} & \cdots & \frac{\partial \alpha_p}{\partial y_p} \end{pmatrix} \begin{pmatrix} c_1^n \\ \vdots \\ c_p^n \end{pmatrix} = 0. \tag{3.43}$$

Since $\frac{\partial \alpha}{\partial y}$ is invertible at every point $y \in \boldsymbol{R}^p$, (3.43) implies that

$$c_1^n = \cdots = c_p^n = 0.$$

Hence the coefficients of $dy_1^{(n-1)}, \cdots, dy_p^{(n-1)}$ reduce to

$$c_1^{n-1}\frac{\partial \alpha_1}{\partial y_1} + \cdots + c_p^{n-1}\frac{\partial \alpha_p}{\partial y_1}, \cdots, c_1^{n-1}\frac{\partial \alpha_1}{\partial y_p} + \cdots + c_p^{n-1}\frac{\partial \alpha_p}{\partial y_p},$$

respectively. Thus similarly $c_1^{n-1} = \cdots = c_p^{n-1} = 0$. Repeating the same calculation, we have

$$c_1^0 = \cdots = c_p^0 = \cdots = c_1^n = \cdots = c_p^n = 0.$$

Thus we have the conclusion. □

**Example 3.5** In this section, we demonstrate that the three-phase permanent-magnet synchronous machine (PMSM) in the case of Y connection [17, 58] is differentially flat. The model of the PMSM is described as

$$F_{\text{pmsm}} := \begin{pmatrix} v_a - r_a i_a - \dot{\psi}_a \\ v_b - r_b i_b - \dot{\psi}_b \\ v_c - r_c i_c - \dot{\psi}_c \\ \psi_a - L_{aa} i_a - L_{ab} i_b - L_{ac} i_c - \psi_m \sin(n_p\theta) \\ \psi_b - L_{ba} i_a - L_{bb} i_b - L_{bc} i_c - \psi_m \sin(n_p\theta - \frac{2\pi}{3}) \\ \psi_c - L_{ca} i_a - L_{cb} i_b - L_{cc} i_c - \psi_m \sin(n_p\theta + \frac{2\pi}{3}) \\ T_e - n_p\psi_m \left\{ (i_a - \frac{1}{2} i_b - \frac{1}{2} i_c)\cos(n_p\theta) + \frac{\sqrt{3}}{2}(i_b - i_c)\sin(n_p\theta) \right\} \\ J\ddot{\theta} - T_e + T_{mec} \\ i_a + i_b + i_c \end{pmatrix} = 0. \tag{3.44}$$

Here each phase of the machine is denoted by "a", "b", and "c". Each variable of the PMSM denotes:

1. voltage across windings: $v_a$, $v_b$, and $v_c$ for phases "a", "b", and "c", respectively.

2. currents through windings: $i_a$, $i_b$, and $i_c$ for phases "a", "b", and "c", respectively.

3. fluxes: $\psi_a$, $\psi_b$, $\psi_c$ for the phases windings.

4. torques: the electromechanical torque produced by the machine $T_e$ and the mechanical torque by the shaft of the machine $T_{\text{mec}}$.

5. the angular position of the rotor with respect to the stator: $\theta$.

Each parameter of the PMSM denotes:

1. winding resistances: $r_a$, $r_b$, and $r_c$ for each phase.

2. winding inductances:

$$L_{xy} = \begin{cases} L_l + L_m & \text{if } x = y, \\ -\frac{1}{2}L_m & \text{if } x \neq y, \end{cases} \qquad x, y \in \{a, b, c\},$$

   where $L_l$ denotes the leakage inductance and $L_m$ the magnetizing inductance.

3. flux: $\psi_m$ for the permanent magnet.
4. the number of pole pairs: $n_p$.
5. rotor inertia: $J$.

Let $(y_1, y_2, y_3) := (\theta, i_b, i_c)$. Then $(y_1, y_2, y_3)$ is a flat output of system (3.44). In fact, by a direct calculation, we get

$$\begin{cases} i_a = -y_2 - y_3, \\ \psi_a = -\left(L_l + \frac{3}{2}L_m\right)(y_2 + y_3) + \psi_m \sin(n_p y_1), \\ \psi_b = \left(L_l + \frac{3}{2}L_m\right) y_2 + \psi_m \sin(n_p y_1 - \frac{2\pi}{3}), \\ \psi_c = \left(L_l + \frac{3}{2}L_m\right) y_3 + \psi_m \sin(n_p y_1 + \frac{2\pi}{3}), \\ v_a = -r_a(y_2 + y_3) - \left(L_l + \frac{3}{2}L_m\right)(\dot{y}_2 + \dot{y}_3) + \psi_m n_p \dot{y}_1 \cos(n_p y_1), \\ v_b = r_b y_2 + \left(L_l + \frac{3}{2}L_m\right)\dot{y}_2 + \psi_m n_p \dot{y}_1 \cos(n_p y_1 - \frac{2\pi}{3}), \\ v_c = r_c y_3 + \left(L_l + \frac{3}{2}L_m\right)\dot{y}_3 + \psi_m n_p \dot{y}_1 \cos(n_p y_1 + \frac{2\pi}{3}), \\ T_e = n_p \psi_m \left\{-\frac{3}{2}(y_2 + y_3)\cos(n_p y_1) + \frac{\sqrt{3}}{2}(y_2 - y_3)\sin(n_p y_1)\right\}, \\ T_{\text{mec}} = n_p \psi_m \left\{-\frac{3}{2}(y_2 + y_3)\cos(n_p y_1) + \frac{\sqrt{3}}{2}(y_2 - y_3)\sin(n_p y_1)\right\} - J\ddot{y}_1. \end{cases} \tag{3.45}$$

Hence condition 1 of definition 3.10 of differential flatness is satisfied. In addition, clearly, condition 2 of definition 3.10 is also satisfied.

Finally, let us consider a smooth map $\alpha : \boldsymbol{R}^3 \to \boldsymbol{R}^3$, $\tilde{y} := (\theta, i_d, i_q) = \alpha(y)$, defined by

$$\begin{cases} \theta = y_1, \\ \begin{pmatrix} i_d \\ i_q \end{pmatrix} = \sqrt{\frac{2}{3}} \begin{pmatrix} \sin(n_p y_1) & \sin(n_p y_1 - \frac{2}{3}\pi) & \sin(n_p y_1 + \frac{2}{3}\pi) \\ \cos(n_p y_1) & \cos(n_p y_1 - \frac{2}{3}\pi) & \cos(n_p y_1 + \frac{2}{3}\pi) \end{pmatrix} \begin{pmatrix} -y_2 - y_3 \\ y_2 \\ y_3 \end{pmatrix} \end{cases}. \tag{3.46}$$

By a straightforward calculation, we have

$$\det\left(\frac{\partial \alpha}{\partial y}\right) = \frac{1}{\sqrt{3}}.$$

Hence $\frac{\partial \alpha}{\partial y}$ is invertible at every point $y \in \boldsymbol{R}^3$. Therefore by theorem 3.5, $(\theta, i_d, i_q)$ is also a flat output of system (3.44). Thus system variables $(v_a, v_b, v_c, \psi_a, \psi_b, \psi_c, T_e, T_{\mathrm{mec}}, \theta, i_a, i_b, i_c)$ can be represented by $(\theta, i_d, i_q)$. Here, $i_d$ and $i_q$ mean currents in fictitious windings rotating at synchronous speed. ■

**Remark 3.7** *In example 3.5, suppose that $r := r_a = r_b = r_c$. Then if we apply the transformation*

$$f_{odq} = \sqrt{\frac{2}{3}} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \sin(n_p\theta) & \sin(n_p\theta - \frac{2}{3}\pi) & \sin(n_p\theta + \frac{2}{3}\pi) \\ \cos(n_p\theta) & \cos(n_p\theta - \frac{2}{3}\pi) & \cos(n_p\theta + \frac{2}{3}\pi) \end{pmatrix} f_{abc} \tag{3.47}$$

*to system (3.44), then system (3.44) is transformed into a simple form*

$$\tilde{F}_{\mathrm{pmsm}} = \begin{pmatrix} i_o \\ v_o - ri_o - L_l \frac{di_o}{dt} \\ \psi_o - L_l i_o \\ v_d - ri_d - \dot{\psi}_d - n_p\dot{\theta}\psi_q \\ v_q - ri_q - \dot{\psi}_q - n_p\dot{\theta}\psi_d \\ \psi_d - (L_l + \frac{3}{2}L_m)i_d - \sqrt{\frac{3}{2}}\psi_m \\ \psi_q - (L_l + \frac{3}{2}L_m)i_q \\ T_e - \sqrt{\frac{3}{2}}n_p\psi_m i_q \\ J\ddot{\theta} - T_e + T_{\mathrm{mec}} \end{pmatrix} = 0, \tag{3.48}$$

*where $f_{abc}$ denotes either $v_{abc}$, $i_{abc}$, or $\psi_{abc}$. Clearly, $(\theta, i_d, i_q)$ is a flat output of system (3.48). Since transformation (3.47), which is called Park transformation [17, 58, 87], is invertible, by a direct calculation, we can check that $(\theta, i_d, i_q)$ is a flat output of system (3.44). However, if $r_a$, $r_b$, and $r_c$ are not equal to each other, we cannot obtain a simple system such as system (3.48) by using transformation (3.47). Nevertheless, even if $r_a$, $r_b$, and $r_c$ are not equal to each other, transformation (3.46) shows that $(\theta, i_d, i_q)$ is a flat output of system (3.44).* ■

## 3.4 Summary

This chapter has given a class of nonlinear differential algebraic systems with geometric index one such that trajectory tracking are easily realized. First, we have defined algebraic controllability and controllable trajectory. As dual concepts, we have introduced algebraic observability and observable trajectory. It has been shown that if a given nonlinear differential algebraic systems with geometric index one is algebraically controllable, every linearized system along any (periodic) controllable trajectory is (uniformly) completely controllable. As a dual result, it has been shown that if a given nonlinear differential algebraic systems with geometric index one is algebraically observable, every linearized system along any (periodic) observable trajectory is (uniformly) completely observable. Finally, for differential algebraic systems which do not distinguish state, input, and output variables, we have studied differential flatness.

# Chapter 4

# Trajectory tracking control of nonlinear systems

As mentioned in interpretation 1 in section 2.2, if a given system is **algebraically controllable** and $(x^*(t), u^*(t))$ is a periodic **controllable trajectory**, we can design a controller $K(t)$ such that the actual trajectory $x(t)$ locally exponentially approaches the reference trajectory $x^*(t)$ by applying a feedforward and state feedback control

$$u(t) = u^*(t) + K(t)(x(t) - x^*(t)) \tag{4.1}$$

into system (2.1)-(2.2). To see this, we should simulate the actual trajectory $x(t)$ of the closed-loop obtained by applying (4.1) into a given system (2.1)-(2.2).

Furthermore, as mentioned in interpretation 2 in section 2.3, if a given system is **algebraically controllable and algebraically observable** and $(x^*(t), u^*(t))$ is a periodic **controllable and observable trajectory**, it is expected that we can design a controller gain $K(t)$ and an observer gain $L(t)$ such that the actual trajectory $x(t)$ locally exponentially approaches the reference trajectory $x^*(t)$ by applying a feedforward and state-estimate feedback control $u(t) = u^*(t) + K(t)\hat{x}_\epsilon$ into system (2.1)-(2.2). To see this, we should simulate

$$\begin{cases} \dot{x} = f(x, u^*(t) + K(t)\hat{x}_\epsilon), \\ x_\epsilon(t) = x(t) - x^*(t), \\ y_\epsilon = C(t)x_\epsilon, \\ \dot{\hat{x}}_\epsilon = A(t)\hat{x}_\epsilon + B(t)K(t)\hat{x}_\epsilon + L(t)(y_\epsilon - C(t)\hat{x}_\epsilon). \end{cases} \tag{4.2}$$

Here, we design a feedback gain $K(t)$ such that the actual trajectory locally exponentially approaches the reference trajectory by applying (4.1) into system (2.1)-(2.2). On the other hand, we must design an observer gain $L(t)$ such that $\hat{x}_\epsilon(t)$ exponentially approaches $x_\epsilon(t)$. To this end, we can use the observer gain proposed in [11]. Concretely, we use

$$L(t) := \gamma\Phi(t, kT)P(t)\Phi^T(t, kT)C^T(t) \tag{4.3}$$

for all $t \in [kT, (k+1)T)$, $T > 0$, $k = 0, 1, 2, \cdots$, where

$$\begin{cases} \dot{P} = -\gamma P \Phi^T(t, kT) C^T(t) C(t) \Phi(t, kT) P, \\ P(kT) = pI > 0, \end{cases}$$

and where $\gamma$ and $p$ are chosen sufficiently large. Here, $\Phi(t, \tau)$ denotes the state transition matrix of the open-loop (2.4).

The aim of this chapter is to demonstrate that trajectory tracking controls of algebraically controllable and observable systems are easily realized.

## 4.1 Trajectory generation

This section explains trajectory generation methods for nonlinear system (2.1). As mentioned in chapter 1, we consider optimal control or flatness-based trajectory generation methods. Although dynamic programming and variational methods as optimal control methods are famous, it is difficult to apply the dynamic programming method for general nonlinear system (2.1) because we have to solve a nonlinear partial differential equation called a Hamilton-Jacobi-Bellman equation [84, 103]. Thus for a trajectory generation, this section only considers a variational method and a flatness-based trajectory generation method.

### 4.1.1 Variational method

This subsection elaborates a trajectory generation based on a variational method. We refer to [43,84,103]. Suppose that $L : \boldsymbol{R}^n \times \boldsymbol{R}^m \to \boldsymbol{R}$ is given. Let us consider

$$\begin{aligned} \min \quad & J := \int_{t_0}^{t_1} L(x(t), u(t)) dt && (4.4) \\ \text{subject to} \quad & (2.1), \\ & x(t_0) = x_0, \quad x(t_1) = x_1. && (4.5) \end{aligned}$$

Note that a linear quadratic optimal control in subsection 2.4.2 is a special case of the above optimal problem. Let

$$\bar{J} := \int_{t_0}^{t_1} \left\{ L(x(t), u(t)) + \lambda^T(t)(f(x(t), u(t)) - \dot{x}(t)) \right\} dt$$

and let

$$H(x, u, \lambda) := L(x, u) + \lambda^T f(x, u).$$

Calculating the first variation of $\bar{J}$, we have

$$
\begin{aligned}
\delta \bar{J} &= \int_{t_0}^{t_1} \left\{ \left( \frac{\partial H}{\partial x} \right)^T \delta x + \left( \frac{\partial H}{\partial u} \right)^T \delta u - \lambda^T \delta \dot{x} \right\} dt \\
&= -[\lambda^T(t)\delta x(t)]_{t_0}^{t_1} + \int_{t_0}^{t_1} \left\{ \left( \frac{\partial H}{\partial x} \right)^T \delta x + \left( \frac{\partial H}{\partial u} \right)^T \delta u + \dot{\lambda}^T(t)\delta x \right\} dt \\
&= \int_{t_0}^{t_1} \begin{pmatrix} \left(\frac{\partial H}{\partial x}\right)^T + \dot{\lambda}^T(t) & \left(\frac{\partial H}{\partial u}\right)^T \end{pmatrix} \begin{pmatrix} \delta x \\ \delta u \end{pmatrix} dt
\end{aligned}
$$

Therefore $\delta \bar{J} = 0$ implies that

$$
\dot{\lambda} = -\frac{\partial H}{\partial x}(x, u, \lambda), \tag{4.6}
$$

$$
\frac{\partial H}{\partial u}(x, u, \lambda) = 0. \tag{4.7}
$$

We say that Eqs. (2.1), (4.5) (4.6), (4.7) are **Euler-Lagrange equations**. Hence we have the following proposition.

**Proposition 4.1** *Suppose that there exists the optimal control input $u(t) \in \boldsymbol{R}^m$, $t_0 \leq t \leq t_1$ such that* (4.4) *is minimized. Let $x(t) \in \boldsymbol{R}^n$ be the corresponding optimal trajectory. Then there exists $\lambda(t) \in \boldsymbol{R}^n$ such that* (2.1), (4.5), (4.6), *and* (4.7) *are satisfied.*

From now on, let us consider solving Eqs. (2.1), (4.5), (4.6), and (4.7) as follows.

1. Give $\lambda(t_0)$.
2. By (4.7), express a control variable $u$ by a state variable $x$ and an adjoint variable $\lambda$.
3. Solve the initial value problem of (2.1) and (4.6).
4. If $||x(t_1) - x_1||$ is sufficiently small, we finish the simulation. If $||x(t_1) - x_1||$ is not sufficiently small, **modify** $\lambda(t_0)$ and return to step 2.

The above numerical procedure is called a **shooting method** [84, 103].

From now on, we give a method of modification of $\lambda(t_0)$. If $\lambda(t_0)$ changed, $x$, $\lambda$, $u$ also change, that is, then $x$, $\lambda$, $u$ become to $x + \delta x$, $\lambda + \delta\lambda$, $u + \delta u$. Furthermore, $x + \delta x$, $\lambda + \delta\lambda$, $u + \delta u$ obey the Euler-Lagrange equations

$$
\begin{cases}
\frac{d}{dt}(x + \delta x) = f(x + \delta x, u + \delta u), \\
\frac{d}{dt}(\lambda + \delta\lambda) = -\frac{\partial H}{\partial x}(x + \delta x, u + \delta u, \lambda + \delta\lambda), \\
\frac{\partial H}{\partial u}(x + \delta x, u + \delta u, \lambda + \delta\lambda) = 0,
\end{cases} \tag{4.8}
$$

where $\delta(x_0) = 0$ and $\delta\lambda(t_0) = \delta\lambda_0$. Let $e$ be sufficiently small positive number. Our goal is to give $\delta\lambda_0$ such that $||x(t_1) - x_1|| < e$. Eq. (4.8) implies that

$$\delta\dot{x} = \frac{\partial f}{\partial x}(x(t), u(t))\delta x + \frac{\partial f}{\partial u}(x(t), u(t))\delta u, \tag{4.9}$$

$$\delta\dot{\lambda} = -\frac{\partial^2 H}{\partial x^2}(x(t), u(t), \lambda(t))\delta - \frac{\partial^2 H}{\partial x \partial u}(x(t), u(t), \lambda(t))\delta u - \frac{\partial^2 H}{\partial x \partial \lambda}(x(t), u(t), \lambda(t))\delta\lambda, \tag{4.10}$$

$$\frac{\partial^2 H}{\partial u \partial x}(x(t), u(t))\delta x + \frac{\partial^2 H}{\partial u^2}(x(t), u(t))\delta u + \frac{\partial^2 H}{\partial u \partial \lambda}(x(t), u(t))\delta\lambda = 0. \tag{4.11}$$

Since

$$\frac{\partial H}{\partial \lambda} = f(x, u),$$

Eq. (4.11) implies that

$$\delta u = -\left(\frac{\partial^2 H}{\partial u^2}\right)^{-1}(x(t), u(t), \lambda(t))\left(\frac{\partial^2 H}{\partial u \partial x}(x(t), u(t), \lambda(t))\delta x + \left(\frac{\partial f}{\partial u}\right)^T(x(t), u(t))\right). \tag{4.12}$$

Substituting (4.12) into (4.9) and (4.10), we have

$$\frac{d}{dt}\begin{pmatrix}\delta x \\ \delta\lambda\end{pmatrix} = \begin{pmatrix} A(t) & -B(t) \\ -C(t) & -A^T(t)\end{pmatrix}\begin{pmatrix}\delta x \\ \delta\lambda\end{pmatrix}, \tag{4.13}$$

where

$$\begin{aligned}
A(t) &:= \frac{\partial f}{\partial x}(x(t), u(t)) - \frac{\partial f}{\partial u}(x(t), u(t))\left(\frac{\partial^2 H}{\partial u^2}\right)^{-1}(x(t), u(t), \lambda(t))\frac{\partial^2 H}{\partial u \partial x}(x(t), u(t), \lambda(t)), \\
B(t) &:= \frac{\partial f}{\partial u}(x(t), u(t))\left(\frac{\partial^2 H}{\partial u^2}\right)^{-1}(x(t), u(t), \lambda(t))\left(\frac{\partial f}{\partial u}\right)^T(x(t), u(t)), \\
C(t) &:= \frac{\partial^2 H}{\partial x^2}(x(t), u(t), \lambda(t)) \\
&\quad - \frac{\partial^2 H}{\partial x \partial u}(x(t), u(t), \lambda(t))\left(\frac{\partial^2 H}{\partial u^2}\right)^{-1}(x(t), u(t), \lambda(t))\frac{\partial^2 H}{\partial u \partial x}(x(t), u(t), \lambda(t)).
\end{aligned}$$

Let $\Phi(t)$ be the transition matrix of (4.13), that is,

$$\frac{d\Phi}{dt} = \begin{pmatrix} A(t) & -B(t) \\ -C(t) & -A^T(t)\end{pmatrix}\Phi, \quad \Phi(t_0) = I.$$

Then if we denote $\Phi$ by $\begin{pmatrix}\Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22}\end{pmatrix}$, we get

$$\begin{pmatrix}\delta x(t) \\ \delta\lambda(t)\end{pmatrix} = \Phi(t)\begin{pmatrix}\delta x(t_0) \\ \delta\lambda(t_0)\end{pmatrix} = \begin{pmatrix}\Phi_{12}(t) \\ \Phi_{22}(t)\end{pmatrix}\delta\lambda_0.$$

Let $E := x(t_1) - x_1$. Then

$$\delta E = \delta x(t_1) = \Phi_{12}(t_1)\delta\lambda_0.$$

If we put

$$\delta E = -pE, \tag{4.14}$$

we have

$$\delta\lambda_0 = -p\left(\Phi_{12}(t_1)\right)^{-1} E, \tag{4.15}$$

where $p \in \boldsymbol{R}_+$ is a parameter.

**Example 4.1** Let us consider an optimal control of system (2.29). Consider

$$\begin{aligned}
\min \quad & \int_0^{t_1} u_1^2(t) + u_2^2(t)dt \qquad (4.16)\\
\text{subject to} \quad & (2.29),\\
& (x(0), y(0), \theta(0)) = (x_0, y_0, \theta_0),\\
& (x(t_1), y(t_1), \theta(t_1)) = (x_1, y_1, \theta_1).
\end{aligned}$$

Then we have the following Euler-Lagrange equation.

$$\begin{aligned}
\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{pmatrix} &= \begin{pmatrix} u_1 \cos\theta \\ u_1 \sin\theta \\ u_2 \end{pmatrix},\\
(x(0), y(0), \theta(0)) &= (x_0, y_0, \theta_0),\\
(x(t_1), y(t_1), \theta(t_1)) &= (x_1, y_1, \theta_1),\\
\dot{\lambda}_3 &= \lambda_1 u_1 \sin\theta - \lambda_2 u_1 \cos\theta,\\
\lambda_1(t) &= \lambda_1(0),\\
\lambda_2(t) &= \lambda_2(0),\\
\begin{pmatrix} 2u_1 + \lambda_1 \cos\theta + \lambda_2 \sin\theta \\ 2u_2 + \lambda_3 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad (4.17)
\end{aligned}$$

Eq. (4.17) implies that

$$u_1 = -\frac{1}{2}(\lambda_1 \cos\theta + \lambda_2 \sin\theta), \tag{4.18}$$

$$u_2 = -\frac{1}{2}\lambda_3 \tag{4.19}$$

Thus using the shooting method, let us solve the initial value problem

$$\begin{aligned}
\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{pmatrix} &= \begin{pmatrix} -\frac{1}{2}\cos\theta(\lambda_1\cos\theta+\lambda_2\sin\theta) \\ -\frac{1}{2}\sin\theta(\lambda_1\cos\theta+\lambda_2\sin\theta) \\ -\frac{1}{2}\lambda_3 \end{pmatrix}, \\
(x(0),y(0),\theta(0)) &= (x_0,y_0,\theta_0), \\
\dot{\lambda}_3 &= \frac{1}{2}(\lambda_2\cos\theta-\lambda_1\sin\theta)(\lambda_1\cos\theta+\lambda_2\sin\theta), \\
\lambda_1(t) &= \lambda_1(0), \\
\lambda_2(t) &= \lambda_2(0), \\
(\lambda_1(0),\lambda_2(0),\lambda_3(0)) &= (\lambda_{1,0},\lambda_{2,0},\lambda_{3,0}).
\end{aligned}$$

By a direct calculation, in the case of the example, we have $A(t)$, $B(t)$, $C(t)$ in (4.13) as follows:

$$A(t) := \begin{pmatrix} 0 & 0 & -\sin\theta(t)u_1(t) - \frac{1}{2}\left(\lambda_1(t)\sin\theta(t)\cos\theta(t) - \lambda_2(t)\cos^2(\theta(t)\right) \\ 0 & 0 & \cos\theta(t)u_1(t) - \frac{1}{2}\left(\lambda_1(t)\sin^2(\theta(t)) - \lambda_2(t)\sin(\theta(t))\cos(\theta(t)\right) \\ 0 & 0 & 0 \end{pmatrix}, \tag{4.20}$$

$$B(t) := \frac{1}{2}\begin{pmatrix} \cos^2(\theta(t)) & \cos(\theta(t))\sin(\theta(t)) & 0 \\ \sin(\theta(t))\cos(\theta(t)) & \sin^2(\theta(t)) & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{4.21}$$

$$C(t) := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & c(t) \end{pmatrix}, \tag{4.22}$$

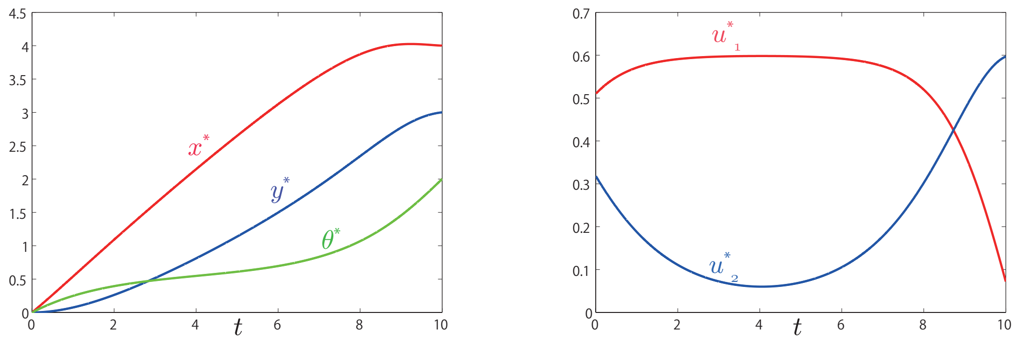
where

$$c(t) := -\lambda_1(t)\cos(\theta(t))u_1(t) - \lambda_2(t)\sin(\theta(t))u_1(t) - (\lambda_1(t)\sin(\theta(t)) - \lambda_2(t)\cos(\theta(t)))^2. \tag{4.23}$$

Let $t_0 = 0$, $t_1 = 10$, $(x_0, y_0, \theta_0) = (0, 0, 0)$, $(x_1, y_1, \theta_1) = (4, 3, 2)$, and $\lambda_0 = (10, 10, 0)$. Then putting the parameter $p$ in (4.15) as 100, we get Fig. 4.1.

However, by using the above method, we **cannot generate** a trajectory which connect $(x_0, y_0, \theta_0) = (0, 0, 0)$ and $(x_1, y_1, \theta_1) = (4, 3, \pi/4)$ because the numerical iterative calculation does not converge. In the next subsection, we demonstrate that if we use a flatness-based trajectory generation technique, we **can generate** such a trajectory. ■

**Example 4.2** Let us generate a periodic trajectory of system (2.29). Consider




(a) The behavior of $(x(t), y(t), \theta(t))$.

(b) Feedforward signals.

Figure 4.1: Trajectory generation

optimal control problem (4.16) and

$$\begin{aligned} \min \quad & \int_{t_1}^{t_2} u_1^2(t) + u_2^2(t)dt \\ \text{subject to} \quad & (2.29), \\ & (x(t_1), y(t_1), \theta(t_1)) = (x_1, y_1, \theta_1), \\ & (x(t_2), y(t_2), \theta(t_2)) = (x_0, y_0, \theta_0). \end{aligned} \tag{4.24}$$

By solving optimal control problems (4.16) and (4.27), we can obtain a periodic trajectory with the period $t_2$ of system (2.29). From now on, let us examine it. Let $t_1 = 4$, $t_2 = 7$, $(x_0, y_0, \theta_0) = (0, 0, 0)$, $(x_1, y_1, \theta_1) = (4, 3, 2)$. Then we can get a trajectory $(x^*(t), y^*(t), \theta^*(t), u_1^*(t), u_2^*(t))$ illustrated in Fig. 4.2.

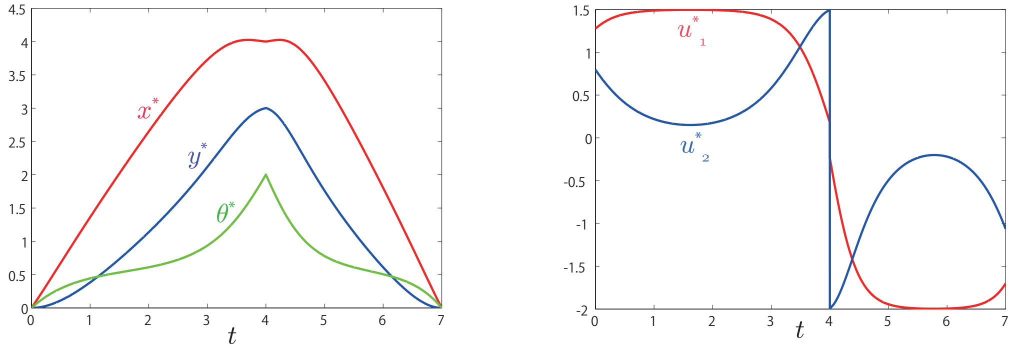
If we apply feedforward control

$$\begin{cases} u_1(t + 7n) = u_1^*(t) \\ u_2(t + 7n) = u_2^*(t) \end{cases} \quad 0 \le t < 7, \ n \in \boldsymbol{Z}, \tag{4.25}$$

and if we take an initial condition $(x_0, y_0, \theta_0) = (0, 0, 0)$, we have a periodic trajectory with the period 7

$$\begin{cases} x(t + 7n) = x^*(t) \\ y(t + 7n) = y^*(t) \\ \theta(t + 7n) = \theta^*(t) \end{cases} \quad 0 \le t < 7, \ n \in \boldsymbol{Z}. \tag{4.26}$$

We write the above periodic trajectory as $(x^*(t), y^*(t), \theta^*(t), u_1^*(t), u_2^*(t))$, again. Although $u_1^*(t)$ and $u_2^*(t)$ are **discontinuous** at $t = n$ and $t = 4n$, $n \in \boldsymbol{Z}$,




(a) The behavior of $(x(t), y(t), \theta(t))$.

(b) Feedforward signals.

Figure 4.2: Periodic trajectory generation

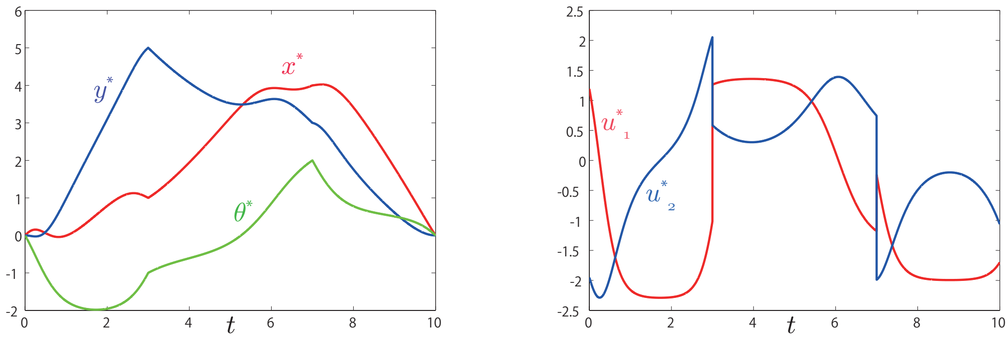
we can consider that $(x^*(t), y^*(t), \theta^*(t), u_1^*(t), u_2^*(t))$ is a periodic controllable and observable trajectory. ■

**Example 4.3** Let us generate a periodic trajectory of system (2.29). Consider optimal control problem (4.16) and the following problems.

$$\begin{aligned} \min \quad & \int_{t_1}^{t_2} u_1^2(t) + u_2^2(t) dt \\ \text{subject to} \quad & (2.29), \\ & (x(t_1), y(t_1), \theta(t_1)) = (x_1, y_1, \theta_1), \\ & (x(t_2), y(t_2), \theta(t_2)) = (x_2, y_2, \theta_2). \end{aligned} \tag{4.27}$$

$$\begin{aligned} \min \quad & \int_{t_2}^{t_3} u_1^2(t) + u_2^2(t) dt \\ \text{subject to} \quad & (2.29), \\ & (x(t_2), y(t_2), \theta(t_2)) = (x_2, y_2, \theta_2), \\ & (x(t_3), y(t_3), \theta(t_3)) = (x_0, y_0, \theta_0). \end{aligned} \tag{4.28}$$

By solving optimal control problems (4.16), (4.27), and (4.28), we can construct a periodic trajectory with the period $t_3$ of system (2.29). From now on, let us examine it. Let $t_1 = 3$, $t_2 = 7$, $t_3 = 10$, $(x_0, y_0, \theta_0) = (0, 0, 0)$, $(x_1, y_1, \theta_1) = (1, 5, -1)$, $(x_2, y_2, \theta_2) = (4, 3, 2)$. Then we can get a trajectory $(x^*(t), y^*(t), \theta^*(t), u_1^*(t), u_2^*(t))$ illustrated in Fig. 4.3.




(a) The behavior of $(x(t), y(t), \theta(t))$. (b) Feedforward signals.

Figure 4.3: Periodic trajectory generation

If we apply feedforward control

$$\begin{cases} u_1(t+10n) = u_1^*(t) \\ u_2(t+10n) = u_2^*(t) \end{cases} \qquad 0 \le t < 10, \ n \in \boldsymbol{Z}, \tag{4.29}$$

and if we take an initial condition $(x_0, y_0, \theta_0) = (0, 0, 0)$, we have a periodic trajectory with the period 10

$$\begin{cases} x(t+10n) = x^*(t) \\ y(t+10n) = y^*(t) \\ \theta(t+10n) = \theta^*(t) \end{cases} \qquad 0 \le t < 10, \ n \in \boldsymbol{Z}. \tag{4.30}$$

We write the above periodic trajectory as $(x^*(t), y^*(t), \theta^*(t), u_1^*(t), u_2^*(t))$, again. Although $u_1^*(t)$ and $u_2^*(t)$ are **discontinuous** at $t = n$, $t = 3n$, and $t = 7n$, $n \in \boldsymbol{Z}$, we can consider that $(x^*(t), y^*(t), \theta^*(t), u_1^*(t), u_2^*(t))$ is a periodic controllable and observable trajectory. ■

### 4.1.2 Flatness-based trajectory generation method

This subsection elaborates a flatness-based trajectory generation method. We refer to [72]. Suppose that system (2.1) is differentially flat with a flat output $v$. Then we can parameterize the components of the flat output $v_i$, $i = 1, \cdots, m$ by

$$v_i(t) := \sum_j a_{ij} \phi_j(t),$$

where $\phi_j(t),\ j = 1, \cdots, N$ are basis functions. **This reduces the problem from finding a function in an infinite dimensional space to finding a finite set of parameters**.

Suppose that we have available to us an initial state $x_0$ at time $t_0$ and a final state $x_1$ at time $t_1$. Then we have

$$\begin{aligned} v_i(t_0) &= \sum_j a_{ij}\phi_j(t_0) & v_i(t_1) &= \sum_j a_{ij}\phi_j(t_1) \\ &\vdots & &\vdots \\ v_i^{(q)}(t_0) &= \sum_j a_{ij}\phi_j^{(q)}(t_0) & v_i^{(q)}(t_1) &= \sum_j a_{ij}\phi_j^{(q)}(t_1) \end{aligned}$$

Therefore to determine $\{a_{ij}\}$, we should solve the following **linear algebraic** equation

$$\begin{pmatrix} \phi_1(t_0) & \phi_3(t_0) & \cdots \\ \phi_1(t_1) & \phi_3(t_1) & \cdots \\ \dot\phi_1(t_0) & \dot\phi_3(t_0) & \cdots \\ \dot\phi_1(t_1) & \dot\phi_3(t_1) & \cdots \\ \vdots & \vdots & \cdots \\ \phi_1^{(q)}(t_0) & \phi_3^{(q)}(t_0) & \cdots \\ \phi_1^{(q)}(t_1) & \phi_3^{(q)}(t_1) & \cdots \end{pmatrix} \begin{pmatrix} a_{11} & \cdots & a_{m1} \\ a_{12} & \cdots & a_{m2} \\ a_{13} & \cdots & a_{m3} \\ \vdots & \cdots & \vdots \end{pmatrix} = \begin{pmatrix} v_1(t_0) & \cdots & v_m(t_0) \\ v_1(t_1) & \cdots & v_m(t_1) \\ \dot v_1(t_0) & \cdots & \dot v_m(t_0) \\ \dot v_1(t_1) & \cdots & \dot v_m(t_1) \\ \vdots & \cdots & \vdots \\ v_1^{(q)}(t_0) & \cdots & v_m^{(q)}(t_0) \\ v_1^{(q)}(t_1) & \cdots & v_m^{(q)}(t_1) \end{pmatrix}.$$

**Example 4.4** Let us consider system (2.29). By a direct calculation from (2.29), we have

$$\begin{cases} \theta = \arctan\left(\frac{\dot y}{\dot x}\right), \\ u_1 = \pm\sqrt{\dot x^2 + \dot y^2}, \\ u_2 = \frac{\ddot y \dot x - \dot y \ddot x}{\dot x^2 + \dot y^2}. \end{cases} \tag{4.31}$$

Hence system (2.29) is differentially flat with a flat output $(x, y)$.

Let $t_0 = 0$ and $t_1 = T$. Now we parameterize $(x, y)$ as follows.

$$\begin{cases} x(t) = a_{11} + a_{12}t + a_{13}t^2 + a_{14}t^3, \\ y(t) = a_{21} + a_{22}t + a_{23}t^2 + a_{24}t^3. \end{cases} \tag{4.32}$$

To determine $a_{11}, \cdots, a_{14}, a_{21}, \cdots, a_{24}$ in (4.32), we solve the following linear equation

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & T & T^2 & T^3 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 2T & 3T^2 \end{pmatrix} \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ a_{13} & a_{23} \\ a_{14} & a_{24} \end{pmatrix} = \begin{pmatrix} x(0) & y(0) \\ x(T) & y(T) \\ \dot x(0) & \dot y(0) \\ \dot x(T) & \dot y(T) \end{pmatrix}.$$

Let $T = 10$, $(x(0), y(0), \theta(0)) = (0,0,0)$, $(x(T), y(T), \theta(T)) = (4, 3, \pi/4)$. Since (4.31) must be satisfied, we assume that $\dot{x}(0) = 0.1$, $\dot{y}(0) = 0$, $\dot{x}(T) = 0.1$, $\dot{y}(T) = 0.1$. Then we have $\theta(0) = 0$ and $\theta(T) = \pi/4$. To determine $a_{11}, \cdots, a_{14}$, $a_{21}, \cdots, a_{24}$ in (4.32), we solve the following linear equation

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 10 & 100 & 1000 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 20 & 300 \end{pmatrix} \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ a_{13} & a_{23} \\ a_{14} & a_{24} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 4 & 3 \\ 0.1 & 0 \\ 0.1 & 0.1 \end{pmatrix}.$$

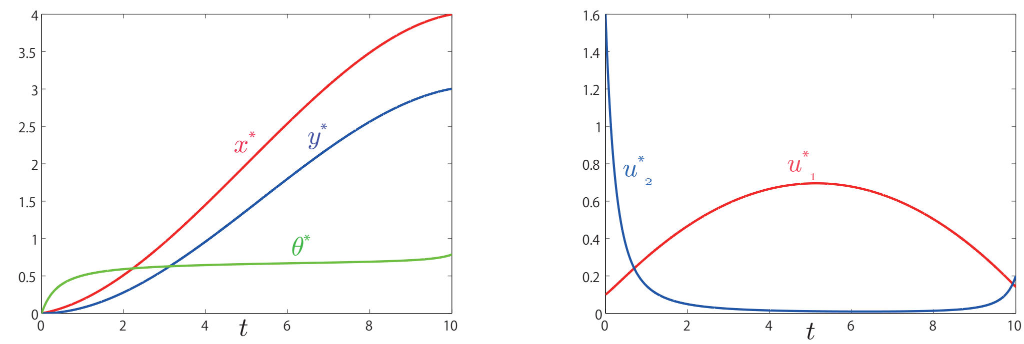
By a direct calculation, we have

$$\begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ a_{13} & a_{23} \\ a_{14} & a_{24} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0.1 & 0 \\ 0.09 & 0.08 \\ -0.006 & -0.005 \end{pmatrix}.$$

Hence from (4.31) we have a feedforward control input

$$\begin{cases} u_1^*(t) = \sqrt{(0.1 + 0.18t - 0.018t^2)^2 + (0.16t - 0.015t^2)^2} \\ u_2^*(t) = \frac{(0.16-0.03t)(0.1+0.18t-0.018t^2)-(0.16t-0.015t^2)(0.18-0.036t)}{(0.1+0.18t-0.018t^2)^2+(0.16t-0.015t^2)^2} \end{cases} \tag{4.33}$$

Under $(x(0), y(0), \theta(0)) = (0,0,0)$, applying the feedforward control (4.33) into system (2.29), we have Fig. 4.4.




(a) The behavior of $(x(t), y(t), \theta(t))$.

(b) Feedforward signals.

Figure 4.4: Trajectory generation

■

If a given system is differentially flat, using the above flatness-based trajectory generation method, we can generate a trajectory by solving linear algebraic equations although if we use variational methods, we have to solve nonlinear ordinary differential equations. Moreover, if a given system is differentially flat and a reference trajectory is given, we can directly calculate an appropriate feedforward control by definition 2.10 of differential flatness.

**Remark 4.1** *Note that the above flatness-based trajectory generation method does not guarantee to generate an optimal trajectory. However, there are some works on a flatness-approach which guarantees optimality [19, 67, 68, 98]. Reference [98] has pointed out that a flatness-approach frequently converts the original convex constraints to non-convex constraints. To resolve the problem, references [67, 68] have studied convex approximations of the non-convex constraints inspired by [19].* ■

## 4.2 Tracking control of algebraically controllable and observable systems

This section shows that a two-degree-of-freedom control is useful for a trajectory tracking control of algebraically controllable and observable systems. For simplicity, the following nonholonomic mobile robot as shown in Fig. 2.1 is studied because the mathematical model is **algebraically controllable and observable** (see examples 2.2 and 2.5).

$$\begin{cases} \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix} u_1 + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u_2, \\ y_1 = x, \\ y_2 = y \end{cases}, \tag{4.34}$$

where $(x, y)$ and $\theta$ denote the wheel-axis-center position and the orientation of the robot, respectively, and $u_1$ and $u_2$ denote the translational and rotational velocities, respectively. Here $(x, y, \theta)$ and $(u_1, u_2)$ denote state and input variables, respectively.

From examples 2.2 and 2.5, any (periodic) trajectory $(x^*(t), y^*(t), \theta^*(t), u_1^*(t), u_2^*(t)) \in (C^\infty_{\mathrm{pw}})^5$ such that $u_1^*(t)$ is bounded on $\boldsymbol{R}$, and $u_1^*(t) \neq 0$ and $\theta^*(t) \neq \frac{n\pi}{2}$ for almost all $t \in \boldsymbol{R}$, $n \in \boldsymbol{Z}$ is a (periodic) **controllable and observable trajectory**. Hence if $(x^*(t), y^*(t), \theta^*(t), u_1^*(t), u_2^*(t))$ is such a trajectory, and if we consider $(x^*(t), y^*(t), \theta^*(t))$ as a reference trajectory, it is expected that we can design a controller such that the actual trajectory $(x(t), y(t), \theta(t))$ locally exponentially approaches the reference trajectory $(x^*(t), y^*(t), \theta^*(t))$. From now on, we examine it.

### 4.2.1 Feedback controller design based on LQ optimal control

In order to design a controller such that the actual trajectory locally exponentially approaches the reference trajectory, let us design a feedback controller based on LQ optimal control explained in subsection 2.4.2.

**Case1:** Let a reference trajectory of system (4.34) be

$$\begin{cases} x^*(t) = \cos(\omega t), \\ y^*(t) = \sin(2\omega t), \\ \theta^*(t) = \arctan\left(-2\frac{\cos(2\omega t)}{\sin(\omega t)}\right). \end{cases} \tag{4.35}$$

As shown in example 4.4, since system (4.34) is differentially flat with a flat output $(x, y)$, we can design a feedforward control. In fact, by relation (4.31), an appropriate feedforward control $(u_1^*(t), u_2^*(t))$ is derived as

$$\begin{cases} u_1^*(t) = \omega\sqrt{\sin^2(\omega t) + 4\cos^2(2\omega t)}, \\ u_2^*(t) = 2\omega\frac{2\sin(2\omega t)\sin(\omega t)+\cos(2\omega t)\cos(\omega t)}{\sin^2(\omega t)+4\cos^2(2\omega t)}. \end{cases} \tag{4.36}$$

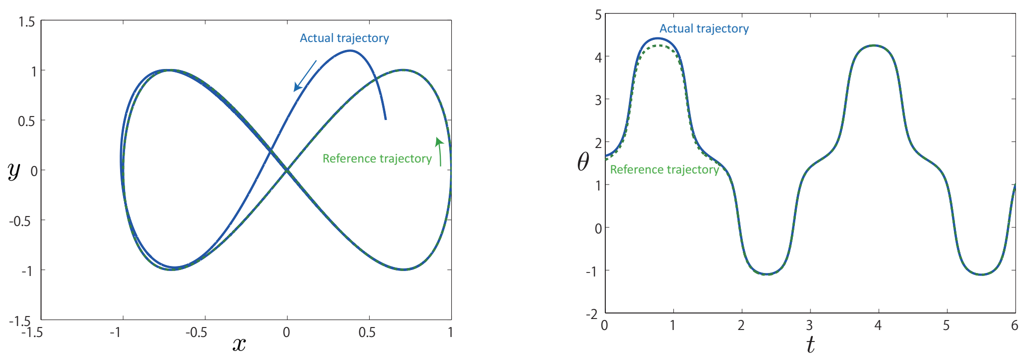
We note that the trajectory $(x^*(t), y^*(t), \theta^*(t), u_1^*(t), u_2^*(t))$ is a **periodic controllable and observable** trajectory. Linearizing system (4.34) along the periodic controllable trajectory $(x^*(t), y^*(t), \theta^*(t), u_1^*(t), u_2^*(t))$, we have

$$\begin{cases} \begin{pmatrix} \dot{x}_\epsilon \\ \dot{y}_\epsilon \\ \dot{\theta}_\epsilon \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 0 & -\sin(\theta^*(t))u_1^*(t) \\ 0 & 0 & \cos(\theta^*(t))u_1^*(t) \\ 0 & 0 & 0 \end{pmatrix}}_{A(t)} \begin{pmatrix} x_\epsilon \\ y_\epsilon \\ \theta_\epsilon \end{pmatrix} + \underbrace{\begin{pmatrix} \cos(\theta^*(t)) & 0 \\ \sin(\theta^*(t)) & 0 \\ 0 & 1 \end{pmatrix}}_{B(t)} \begin{pmatrix} u_{1,\epsilon} \\ u_{2,\epsilon} \end{pmatrix}, \\ \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}}_{C} \begin{pmatrix} x_\epsilon \\ y_\epsilon \end{pmatrix}. \end{cases} \tag{4.37}$$

By theorems 2.6 and 2.11, linearized system (4.37) is completely controllable and completely observable. Thus the Riccati equation (2.61) has the unique positive definite periodic solution [3, 4]. We put $R(t)$ in (2.59) as $R(t) = \frac{1}{5}I_2$. Let $\omega := 2$. Note that system (4.37) has the period $\pi$. If we apply $u_\epsilon = -5B^T(t)P(t)\begin{pmatrix} x_\epsilon \\ y_\epsilon \\ \theta_\epsilon \end{pmatrix}$ into (4.37), the origin of the resulting closed-loop is exponentially stable [3, 4]. Then as mentioned in interpretation 2.3, if we apply a feedforward and feedback control

$$u(t) = u^*(t) + \left( -5B^T(t)P(t) \begin{pmatrix} x(t) - x^*(t) \\ y(t) - y^*(t) \\ \theta(t) - \theta^*(t) \end{pmatrix} \right) \tag{4.38}$$

into system (4.34), the actual trajectory $(x(t), y(t), \theta(t))$ locally exponentially approaches the reference trajectory $(x^*(t), y^*(t), \theta^*(t))$. To see this, applying (4.38) into system (4.34), we simulate the actual trajectory $(x(t), y(t), \theta(t))$ of the resulting closed-loop. Fig. 4.5 illustrates the behavior of the resulting closed-loop under $(x(0), y(0), \theta(0)) = (0.6, 0.5, \frac{\pi}{2} + 0.1)$ although $(x^*(0), y^*(0), \theta^*(0)) = (1, 0, \pi/2)$. We can see that the actual trajectory $(x(t), y(t), \theta(t))$ exponentially approaches the reference trajectory $(x^*(t), y^*(t), \theta^*(t))$.




(a) The behavior of $(x(t), y(t))$.

(b) The behavior of $\theta(t)$.

Figure 4.5: The behavior of the resulting closed-loop

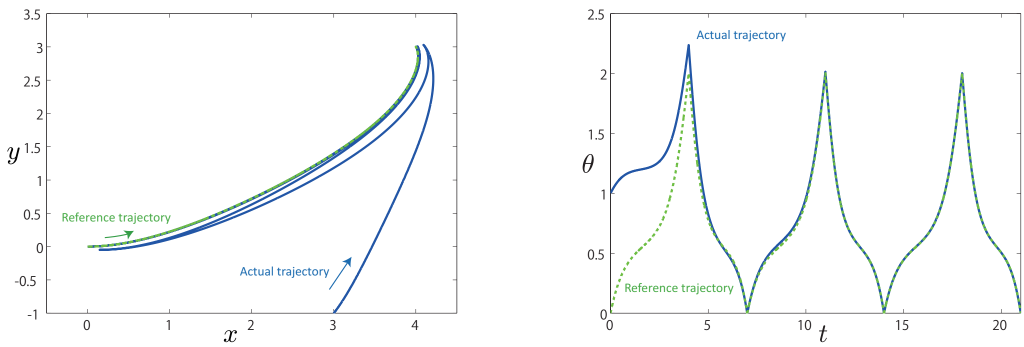
On the other hand, if $\omega := 1$, system (4.41) has the period $2\pi$. Unfortunately, by using the periodic generator method explained in subsection 2.4.2, we cannot numerically solve the Riccati equation (2.61) because the method is highly sensitive due to the ill-conditioning of linear Hamiltonian ODE (2.62). Thus in general, we cannot use the periodic generator method in the case of a large period. If we have to solve the Riccati equation (2.61) which has a large period, we should use another numerical method such as the **multiple shooting method** [27,110].

**Case 2:** Let a reference trajectory of system (4.34) be (4.26). Assume that $(x^*(t), y^*(t), \theta^*(t), u_1^*(t), u_2^*(t))$ is defined by (4.25)-(4.26). Then as mentioned in example 4.2, $(x^*(t), y^*(t), \theta^*(t), u_1^*(t), u_2^*(t))$ is **periodic controllable and observable** trajectory of system (4.34). Linearizing system (4.34) along the periodic controllable trajectory $(x^*(t), y^*(t), \theta^*(t), u_1^*(t), u_2^*(t))$, we have (4.37).

By theorems 2.6 and 2.11, linearized system (4.37) is completely controllable and completely observable. Thus the Riccati equation (2.61) has the unique positive definite periodic solution [3, 4]. We put $R(t)$ in (2.59) as $R(t) = 10I_2$. Then as mentioned in interpretation 2.3, if we apply a feedforward and feedback control

$$u(t) = u^*(t) + \left( -\frac{1}{10}B^T(t)P(t) \begin{pmatrix} x(t) - x^*(t) \\ y(t) - y^*(t) \\ \theta(t) - \theta^*(t) \end{pmatrix} \right) \tag{4.39}$$

into system (4.34), the actual trajectory $(x(t), y(t), \theta(t))$ locally exponentially approaches the reference trajectory $(x^*(t), y^*(t), \theta^*(t))$. To see this, applying (4.39) into system (4.34), we simulate the actual trajectory $(x(t), y(t), \theta(t))$ of the resulting closed-loop. Fig. 4.6 illustrates the behavior of the resulting closed-loop under $(x(0), y(0), \theta(0)) = (3, -1, 1)$ although $(x^*(0), y^*(0), \theta^*(0)) = (0, 0, 0)$. We can see that the actual trajectory $(x(t), y(t), \theta(t))$ exponentially approaches the reference trajectory $(x^*(t), y^*(t), \theta^*(t))$.




(a) The behavior of $(x(t), y(t))$.

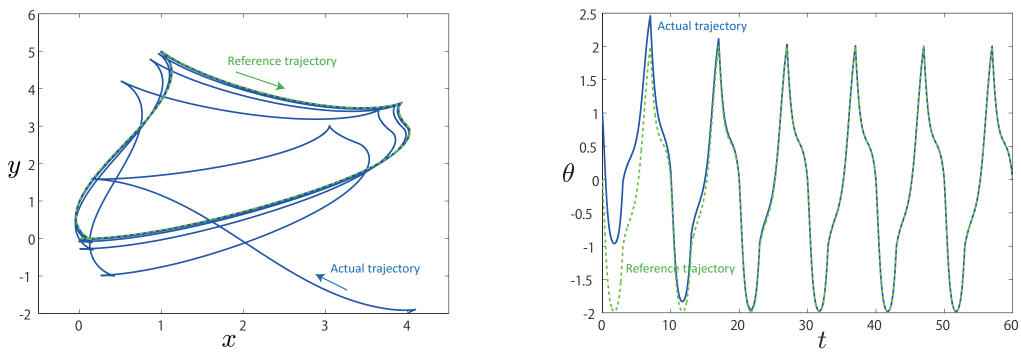
(b) The behavior of $\theta(t)$.

Figure 4.6: The behavior of the resulting closed-loop

**Case 3:** Let a reference trajectory of system (4.34) be (4.30). Assume that $(x^*(t), y^*(t), \theta^*(t), u_1^*(t), u_2^*(t))$ is defined by (4.29)-(4.30). Then as mentioned in example 4.3, $(x^*(t), y^*(t), \theta^*(t), u_1^*(t), u_2^*(t))$ is **periodic controllable and observable** trajectory of system (4.34). Linearizing system (4.34) along the periodic controllable trajectory $(x^*(t), y^*(t), \theta^*(t), u_1^*(t), u_2^*(t))$, we have (4.37). By theorems 2.6 and 2.11, linearized system (4.37) is completely controllable and completely observable. Thus the Riccati equation (2.61) has the unique positive definite periodic solution [3, 4]. We put $R(t)$ in (2.59) as $R(t) = 140I_2$. Then as mentioned in interpretation 2.3, if we apply a feedforward and feedback control

$$u(t) = u^*(t) + \left( -\frac{1}{140} B^T(t) P(t) \begin{pmatrix} x(t) - x^*(t) \\ y(t) - y^*(t) \\ \theta(t) - \theta^*(t) \end{pmatrix} \right) \tag{4.40}$$

into system (4.34), the actual trajectory $(x(t), y(t), \theta(t))$ locally exponentially approaches the reference trajectory $(x^*(t), y^*(t), \theta^*(t))$. To see this, applying (4.40) into system (4.34), we simulate the actual trajectory $(x(t), y(t), \theta(t))$ of the resulting closed-loop. Fig. 4.7 illustrates the behavior of the resulting closed-loop under $(x(0), y(0), \theta(0)) = (4, -2, 1)$ although $(x^*(0), y^*(0), \theta^*(0)) = (0, 0, 0)$. We can see that the actual trajectory $(x(t), y(t), \theta(t))$ exponentially approaches the reference trajectory $(x^*(t), y^*(t), \theta^*(t))$.




(a) The behavior of $(x(t), y(t))$. (b) The behavior of $\theta(t)$.

Figure 4.7: The behavior of the resulting closed-loop

### 4.2.2 Feedback controller design based on LMI

As mentioned in subsection 4.2.1, if linearized system (4.37) has a large period, it is difficult to apply a feedback controller design based on LQ optimal control. However, we can also use a **linear matrix inequality (LMI)** technique [6] to design a stabilizing feedback controller. For system (4.37), it is not easy to apply an LMI technique. Thus we transform a coordinate of (4.37) such as

$$\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} \cos(\theta^*(t)) & \sin(\theta^*(t)) & 0 \\ -\sin(\theta^*(t)) & \cos(\theta^*(t)) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_\epsilon \\ y_\epsilon \\ \theta_\epsilon \end{pmatrix}.$$

Then system (4.37) is transformed into

$$\dot{e} = \underbrace{\begin{pmatrix} 0 & u_2^*(t) & 0 \\ -u_2^*(t) & 0 & u_1^*(t) \\ 0 & 0 & 0 \end{pmatrix}}_{A(t)} e + \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}}_{B} u_\epsilon. \tag{4.41}$$

We note that linear system (4.41) has been derived from (4.34) in example 2.9. If we apply a feedback control $u_\epsilon = K(t)e$ to system (4.41), then we have the closed-loop

$$\dot{e} = (A(t) + BK(t))e. \tag{4.42}$$

In example 2.9, we have shown that system (4.41) is uniformly completely controllable if $(x^*(t), y^*(t), \theta^*(t), u_1^*(t), u_2^*(t))$ is a periodic controllable trajectory. Hence as mentioned in section 2.1, we can design a feedback gain $K(t)$ such that $e = 0$ of (4.42) is exponentially stable. For simplicity, let us consider to find a constant gain $K$ such that $e = 0$ of (4.42) is exponentially stable instead of a time varying gain $K(t)$

Let $P \in \boldsymbol{R}^{3\times 3}$ be a positive definite symmetric matrix. To analyze stability of the origin of system (4.42), we introduce

$$V(e) := e^T P e. \tag{4.43}$$

The derivative of $V(e)$ along the trajectories of (4.42) is given by

$$\dot{V}\big|_{(4.42)}(t,e) := \frac{\partial V}{\partial e}(A(t) + BK)e = 2e^T P(A(t) + BK)e.$$

If there exists an $r > 0$ such that

$$\dot{V}\big|_{(4.42)}(t,e) \leq -2rV(e) \tag{4.44}$$

for all $e \in \boldsymbol{R}^3$ and for all $t \geq 0$, the origin of system (4.42) is exponentially stable [48]. Clearly, we have the following sufficient condition for (4.44) to hold.

**Lemma 4.1** *Suppose that $r > 0$ is given. If there exist $0 < P \in \boldsymbol{R}^{3\times 3}$ and $K \in \boldsymbol{R}^{2\times 3}$ such that*

$$P(A(t) + BK) + (A(t) + BK)^T P \leq -2rP \quad \text{for all} \quad t \geq 0, \tag{4.45}$$

*then* (4.44) *holds.*

In the above lemma, (4.45) is not a linear matrix inequality (LMI). Multiplying $\tilde{P} := P^{-1}$ from the left and right of (4.45), we have an LMI condition for (4.44) to hold as follows.

**Lemma 4.2** *Suppose that $r > 0$ is given. If there exist $0 < \tilde{P} \in \boldsymbol{R}^{3\times 3}$ and $Y \in \boldsymbol{R}^{2\times 3}$ such that*

$$A(t)\tilde{P} + BY + \tilde{P}A^T(t) + Y^T B^T \le -2r\tilde{P} \quad \text{for all} \quad t \ge 0, \tag{4.46}$$

*then* (4.44) *holds.*

In order to numerically check whether or not (4.46) holds, we relax the infinite number of LMI constraints into a finite number of LMI constraints. Suppose that $u_i^*(t)$, $i = 1, 2$ are bounded for all $t \ge 0$. Then there exist $u_{i,\inf}^*$ and $u_{i,\sup}^*$, $i = 1, 2$ such that $u_{i,\inf}^* = \inf\{u_i^*(t) \mid t \ge 0\}$ and $u_{i,\sup}^* = \sup\{u_i^*(t) \mid t \ge 0\}$. Hence we have

$$\begin{aligned} \begin{pmatrix} u_1^*(t) \\ u_2^*(t) \end{pmatrix} &= \lambda_1(t) \begin{pmatrix} u_{1,\inf}^* \\ u_{2,\inf}^* \end{pmatrix} + \lambda_2(t) \begin{pmatrix} u_{1,\inf}^* \\ u_{2,\sup}^* \end{pmatrix} \\ &\quad + \lambda_3(t) \begin{pmatrix} u_{1,\sup}^* \\ u_{2,\inf}^* \end{pmatrix} + \lambda_4(t) \begin{pmatrix} u_{1,\sup}^* \\ u_{2,\sup}^* \end{pmatrix}, \end{aligned} \tag{4.47}$$

where $\lambda_i(t) \ge 0$, $i = 1, \cdots, 4$, and $\lambda_1(t) + \cdots \lambda_4(t) = 1$. Hence we obtain

$$A(t) = \lambda_1(t) \underbrace{\begin{pmatrix} 0 & u_{2,\inf}^* & 0 \\ -u_{2,\inf}^* & 0 & u_{1,\inf}^* \\ 0 & 0 & 0 \end{pmatrix}}_{A_1} + \lambda_2(t) \underbrace{\begin{pmatrix} 0 & u_{2,\sup}^* & 0 \\ -u_{2,\sup}^* & 0 & u_{1,\inf}^* \\ 0 & 0 & 0 \end{pmatrix}}_{A_2} \tag{4.48}$$

$$+ \lambda_3(t) \underbrace{\begin{pmatrix} 0 & u_{2,\inf}^* & 0 \\ -u_{2,\inf}^* & 0 & u_{1,\sup}^* \\ 0 & 0 & 0 \end{pmatrix}}_{A_3} + \lambda_4(t) \underbrace{\begin{pmatrix} 0 & u_{2,\sup}^* & 0 \\ -u_{2,\sup}^* & 0 & u_{1,\sup}^* \\ 0 & 0 & 0 \end{pmatrix}}_{A_4}. \tag{4.49}$$

Since

$$A_i\tilde{P} + BY + \tilde{P}A_i^T + Y^T B^T \le -2r\tilde{P}, \; i = 1, \cdots, 4 \tag{4.50}$$

are a sufficient condition for (4.46) to hold, we have the following theorem:

**Theorem 4.2** *Suppose that $u_i^*(t)$, $i = 1, 2$ are bounded on $\boldsymbol{R}$. Let $u_{i,\inf}^* := \inf\{u_i^*(t) \mid t \ge 0\}$ and $u_{i,\sup}^* := \sup\{u_i^*(t) \mid t \ge 0\}$. Moreover, suppose that $r > 0$ is given. If there exist $0 < \tilde{P} \in \boldsymbol{R}^{3\times 3}$ and $Y \in \boldsymbol{R}^{2\times 3}$ such that* (4.50) *holds, then the feedback gain $K := Y\tilde{P}^{-1}$ exponentially stabilizes the origin of system* (4.42).

## Simulations: State feedback

By solving LMIs (4.50), we can obtain a feedback gain $K$ such that the origin of closed-loop (4.42) is exponentially stable. Then as mentioned in subsection 2.5, if we apply a feedforward and feedback control

$$u(t) = u^*(t) + Ke \tag{4.51}$$

into system (4.34), the actual trajectory $(x(t), y(t), \theta(t))$ locally exponentially approaches the reference trajectory $(x^*(t), y^*(t), \theta^*(t))$. To see this, applying (4.51) into system (4.34), we simulate the actual trajectory $(x(t), y(t), \theta(t))$ of the resulting closed-loop. In this case, we should simulate

$$\begin{cases} \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & 0 \\ \sin\theta & 0 \\ 0 & 1 \end{pmatrix} (u^*(t) + Ke)\,, \\ e(t) = \begin{pmatrix} \cos(\theta(t)) & \sin(\theta(t)) & 0 \\ -\sin(\theta(t)) & \cos(\theta(t)) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x(t) - x^*(t) \\ y(t) - y^*(t) \\ \theta(t) - \theta^*(t) \end{pmatrix}. \end{cases} \tag{4.52}$$

**Case 1:** Let a reference trajectory of system (4.34) be (4.35). Then an appropriate feedforward control $(u_1^*(t), u_2^*(t))$ is derived as (4.36). Since

$$\begin{cases} 1.3 \le u_1^*(t) \le 4.5, \\ -12 \le u_2^*(t) \le 12 \end{cases}$$

on $\boldsymbol{R}$, we put $u_{1,\mathrm{inf}}^* = 1.3$, $u_{1,\mathrm{sup}}^* = 4.5$, $u_{2,\mathrm{inf}}^* = -12$, $u_{2,\mathrm{sup}}^* = 12$. Then under $r = 0.4$, we got a feedback gain

$$K = \begin{pmatrix} -5.6132 & 0 & 0 \\ 0 & -6.5089 & -9.9614 \end{pmatrix}. \tag{4.53}$$

Fig. 4.8 illustrates the behavior of closed-loop (4.52) by using the feedback gain (4.53) under $(x(0), y(0), \theta(0)) = (0.6, 0.5, \frac{\pi}{2} + 0.1)$ although $(x^*(0), y^*(0), \theta^*(0)) = (1, 0, \pi/2)$. We can see that the actual trajectory $(x(t), y(t), \theta(t))$ exponentially approaches the reference trajectory $(x^*(t), y^*(t), \theta^*(t))$. Moreover, Fig. 4.9 illustrates feedback signals of the LQ optimal control and the LMI methods. We can see that the LMI method uses large signals compared to the LQ optimal control method.

Next, let $\omega := 0.34$. Then since linearized system (4.37) has a large period, by using the periodic generator method, we cannot design a feedback controller based on LQ optimal control. However, we can design a feedback controller based on LMI. In fact, since

$$\begin{cases} 0.2 \le u_1^*(t) \le 0.8, \\ -2 \le u_2^*(t) \le 2 \end{cases}$$

on $\boldsymbol{R}$, we put $u^*_{1,\text{inf}} = 0.2$, $u^*_{1,\text{sup}} = 0.8$, $u^*_{2,\text{inf}} = -2$, $u^*_{2,\text{sup}} = 2$. Then under $r = 0.25$, we got a feedback gain

$$K = \begin{pmatrix} -14.6894 & 0 & 0 \\ 0 & -267.9342 & -41.7846 \end{pmatrix}. \tag{4.54}$$

Fig. 4.10 illustrates the behavior of closed-loop (4.52) by using the feedback gain (4.54) under $(x(0), y(0), \theta(0)) = (-0.65, 2.79, -1.30)$ although $(x^*(0), y^*(0), \theta^*(0)) = (1, 0, \pi/2)$.

We can see that the actual trajectory $(x(t), y(t), \theta(t))$ exponentially approaches the reference trajectory $(x^*(t), y^*(t), \theta^*(t))$.

**Case 2:** Let a reference trajectory of system (4.34) be

$$\begin{cases} x^*(t) = \sin(2\omega t), \\ y^*(t) = \sin(3\omega t), \\ \theta^*(t) = \arctan\left(\frac{3\cos(3\omega t)}{2\cos(2\omega t)}\right), \end{cases} \tag{4.55}$$

where $\omega := 0.2$. By relation (4.31), an appropriate feedforward control $(u_1^*(t), u_2^*(t))$ is derived as

$$\begin{cases} u_1^*(t) = \omega\sqrt{4\cos^2(2\omega t) + 9\cos^2(3\omega t)}, \\ u_2^*(t) = 6\omega\frac{2\sin(2\omega t)\cos(3\omega t) - 3\sin(3\omega t)\cos(2\omega t)}{4\cos^2(2\omega t) + 9\cos^2(3\omega t)}. \end{cases} \tag{4.56}$$

We note that the trajectory $(x^*(t), y^*(t), \theta^*(t), u_1^*(t), u_2^*(t))$ is a **periodic controllable and observable** trajectory. Since

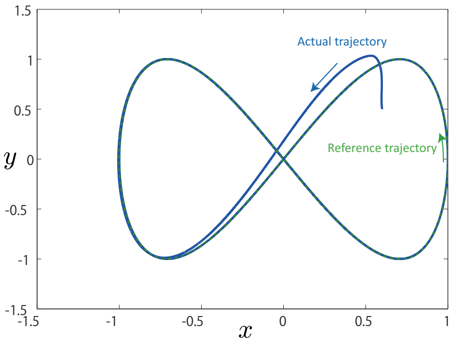
$$\begin{cases} 0.18 \le u_1^*(t) \le 0.73, \\ -2.1 \le u_2^*(t) \le 0 \end{cases}$$

on $\boldsymbol{R}$, we put $u_{1,\text{inf}} = 0.18$, $u_{1,\text{max}} = 0.73$, $u_{2,\text{inf}} = -2.1$, $u_{2,\text{sup}} = 0$. Then under $r = 0.25$, we got a feedback gain

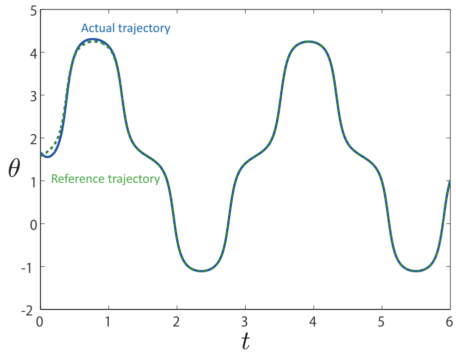
$$K = \begin{pmatrix} -3.4692 & -3.7711 & -1.4439 \\ -4.4038 & -11.0112 & -5.2592 \end{pmatrix}. \tag{4.57}$$

We note that the form of $K$ does not correspond with that of a feedback gain proposed in reference [51]. Fig. 4.11 illustrates the behavior of closed-loop (4.52) by using the feedback gain (4.57) under $(x(0), y(0), \theta(0)) = (0.8804, -4.1954, -0.7053)$ although $(x^*(0), y^*(0), \theta^*(0)) = (0, 0, 0.9828)$.

We can see that the actual trajectory $(x(t), y(t), \theta(t))$ exponentially approaches the reference trajectory $(x^*(t), y^*(t), \theta^*(t))$.




(a) The behavior of $(x(t), y(t))$.




(b) The behavior of $\theta(t)$.

Figure 4.8: The behavior of the resulting closed-loop.

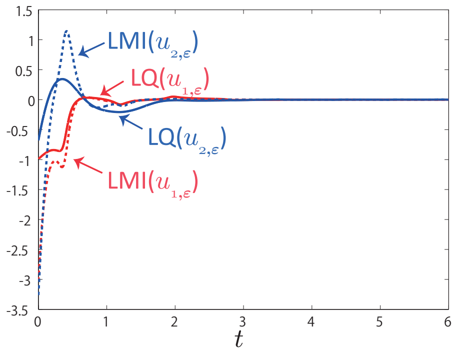
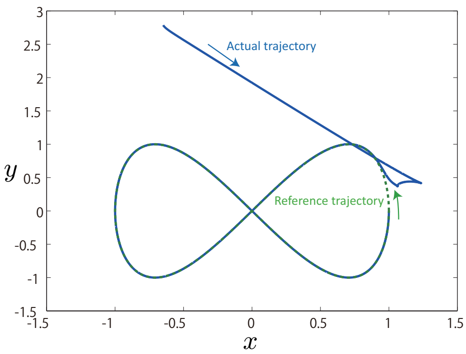
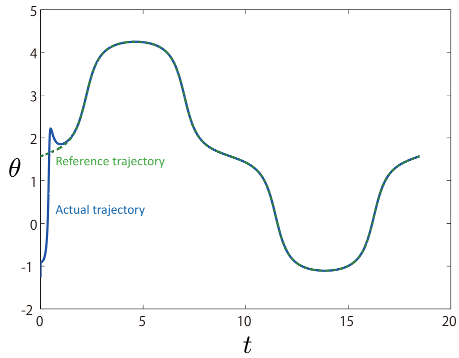



Figure 4.9: Feedback signals of LQ optimal control and LMI methods.

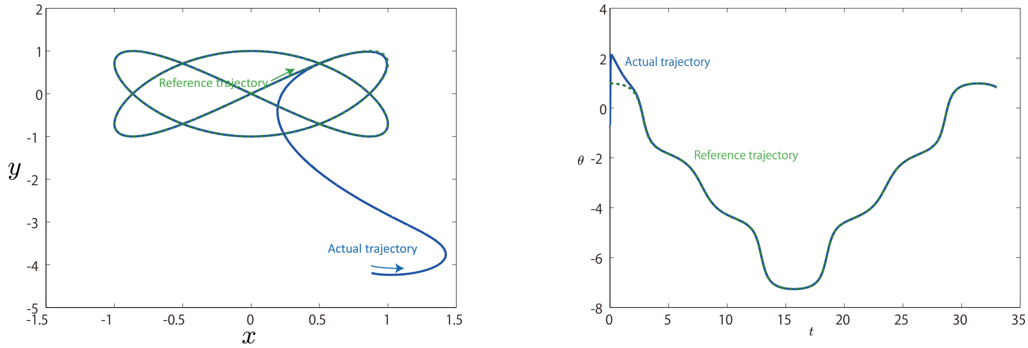



(a) The behavior of $(x(t), y(t))$.




(b) The behavior of $\theta(t)$.

Figure 4.10: The behavior of the resulting closed-loop.




(a) The behavior of $(x(t), y(t))$. (b) The behavior of $\theta(t)$.

Figure 4.11: The behavior of the resulting closed-loop.

**Case 3:** Let us go back to example 4.4. That is, let a reference trajectory of system (4.34) be

$$\begin{cases} x^*(t) = 0.1t + 0.09t^2 - 0.006t^3, \\ y^*(t) = 0.08t^2 - 0.005t^3, \\ \theta^*(t) = \arctan\left(\frac{0.16t - 0.015t^2}{0.1 + 0.18t - 0.018t^2}\right). \end{cases} \tag{4.58}$$

on $[0, 10]$. The corresponding feedforward control is given by (4.33). We note that the trajectory $(x^*(t), y^*(t), \theta^*(t), u_1^*(t), u_2^*(t))$ is a **controllable and observable** trajectory if the trajectory is defined on $\boldsymbol{R}$.

In contrast to the cases 1 and 2, the reference trajectory of this case has been only defined on $[0, 10]$. To apply theorem 4.2, we consider that $u_1^*(t)$ and $u_2^*(t)$ are defined on $\boldsymbol{R}$ and

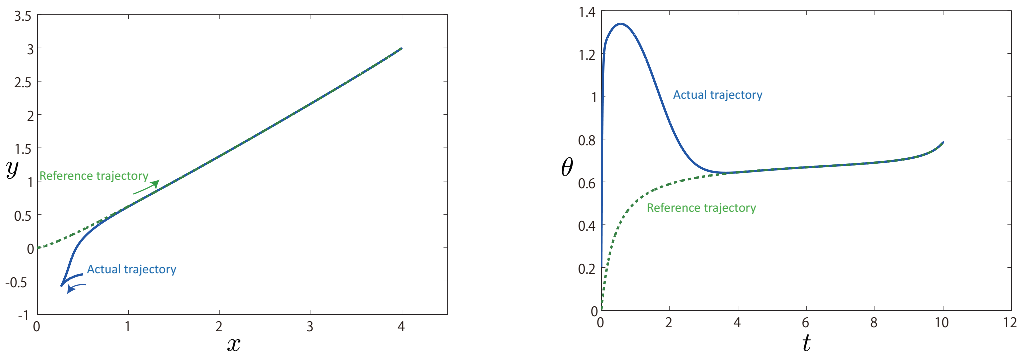
$$\begin{cases} 0.1 \le u_1^*(t) \le 0.7, \\ 0 \le u_2^*(t) \le 1.6 \end{cases} \tag{4.59}$$

on $\boldsymbol{R}$ although actually (4.59) is satisfied only on $[0, 10]$. Thus, we put $u_{1,\mathrm{inf}}^* = 0.1$, $u_{1,\mathrm{sup}}^* = 0.7$, $u_{2,\mathrm{inf}}^* = 0$, $u_{2,\mathrm{sup}}^* = 1.6$. Then under $r = 0.25$, we got a feedback gain

$$K = \begin{pmatrix} -6.8261 & 17.8238 & 4.8280 \\ 11.8126 & -49.9347 & -14.9373 \end{pmatrix}. \tag{4.60}$$

Fig. 4.12 illustrates the behavior of closed-loop (4.52) by using the feedback gain (4.60) under $(x(0), y(0), \theta(0)) = (0.5, -0.4, 0.2)$ although $(x^*(0), y^*(0), \theta^*(0)) = (0, 0, 0)$.

We can see that the actual trajectory $(x(t), y(t), \theta(t))$ exponentially approaches the reference trajectory $(x^*(t), y^*(t), \theta^*(t))$.




(a) The behavior of $(x(t), y(t))$.

(b) The behavior of $\theta(t)$.

Figure 4.12: The behavior of the resulting closed-loop.

## Simulations: State-estimate feedback

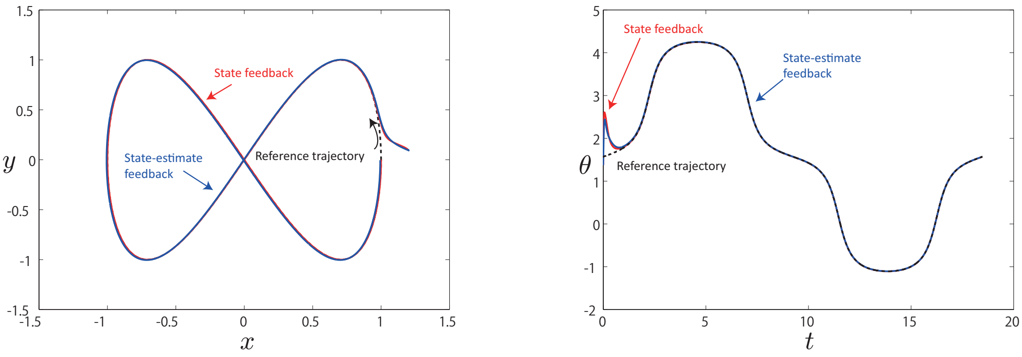
If the available signal in system (4.34) is only output signal $(y_1, y_2)$, we cannot use the control (4.51). Alternatively we need to design an appropriate state observer. In this case, to see exponential stability of the reference trajectory $(x^*(t), y^*(t), \theta^*(t))$, we should simulate

$$\begin{cases} \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & 0 \\ \sin\theta & 0 \\ 0 & 1 \end{pmatrix} (u^*(t) + K\hat{e}), \\ e(t) = \begin{pmatrix} \cos(\theta(t)) & \sin(\theta(t)) & 0 \\ -\sin(\theta(t)) & \cos(\theta(t)) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x(t) - x^*(t) \\ y(t) - y^*(t) \\ \theta(t) - \theta^*(t) \end{pmatrix}, \\ \begin{pmatrix} y_{1,\epsilon} \\ y_{2,\epsilon} \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}}_{C} e, \\ \dot{\hat{e}} = (A(t) + BK)\hat{e} + L(t)\left(\begin{pmatrix} y_{1,\epsilon} \\ y_{2,\epsilon} \end{pmatrix} - C\hat{e}\right), \end{cases} \tag{4.61}$$

where $L(t)$ is defined by (4.3).

**Case 1:** Let a reference trajectory of system (4.34) be (4.35), where $\omega := 0.34$. By relation (4.31), an appropriate feedforward control $(u_1^*(t), u_2^*(t))$ is given by (4.36). In this case, we have a feedback gain (4.54). Let $T = 20$, $\gamma = 50$, $p = 10$. Fig. 4.13

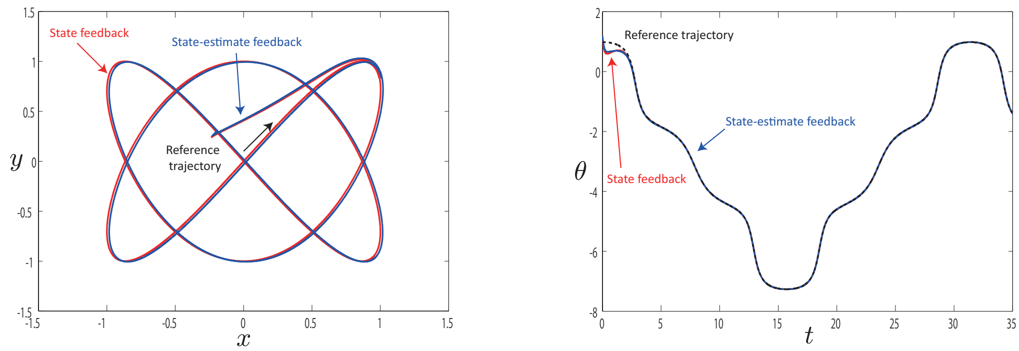
illustrates the behavior of closed-loop (4.61) by using the feedback gain (4.54) under $(x(0), y(0), \theta(0)) = (1.2, 0.1, \pi/2 - 0.2)$ although $(x^*(0), y^*(0), \theta^*(0)) = (1, 0, \pi/2)$.




(a) The behavior of $(x(t), y(t))$. (b) The behavior of $\theta(t)$.

Figure 4.13: The behavior of the resulting closed-loop.

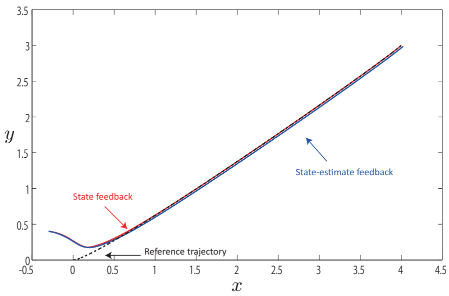
**Case 2:** Let a reference trajectory of system (4.34) be (4.55), where $\omega := 0.2$. By relation (4.31), an appropriate feedforward control $(u_1^*(t), u_2^*(t))$ is given by (4.56). In this case, we have a feedback gain (4.57). Fig. 4.14 illustrates the behavior of closed-loop (4.61) by using the feedback gain (4.57) under $(x(0), y(0), \theta(0)) = (-0.2, 0.3, 0.7828)$ although $(x^*(0), y^*(0), \theta^*(0)) = (0, 0, 0.9828)$.



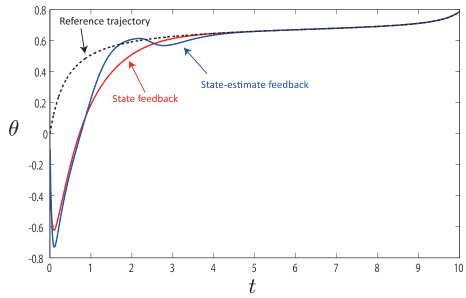

(a) The behavior of $(x(t), y(t))$. (b) The behavior of $\theta(t)$.

Figure 4.14: The behavior of the resulting closed-loop.

**Case 3:** Let a reference trajectory of system (4.34) be (4.58) on $[0, 10]$. By relation (4.31), an appropriate feedforward control $(u_1^*(t), u_2^*(t))$ is given by (4.33). In this case, we have a feedback gain (4.60). Fig. 4.15 illustrates the behavior of closed-loop (4.61) by using the feedback gain (4.60) under $(x(0), y(0), \theta(0)) = (-0.3, -0.4, -0.1)$ although $(x^*(0), y^*(0), \theta^*(0)) = (0, 0, 0)$.




(a) The behavior of $(x(t), y(t))$.




(b) The behavior of $\theta(t)$.

Figure 4.15: The behavior of the resulting closed-loop.

## 4.3 Tracking control of algebraically controllable and observable DAS

This section shows that a two-degree-of-freedom control is useful for a trajectory tracking control of algebraically controllable and observable DAS with geometric index For simplicity, the following simple circuit model as shown in Fig. 3.1 is studied because the mathematical model is **algebraically controllable and observable** (see examples 3.3 and 3.4).

$$\begin{cases} L_1 \frac{di_1}{dt} = -e + u, \\ L_2 \frac{di_2}{dt} = e, \\ 0 = ce + I_0(\exp(ke) - 1) + i_2 - i_1, \\ y = i_1, \end{cases} \tag{4.62}$$

From examples 3.3 and 3.4, any (periodic)trajectory $(i_1^*(t), i_2^*(t), e^*(t), u^*(t)) \in (C_{\mathrm{pw}}^\infty)^4$ of system (4.62) such that $\exp(ke^*(t))$ is bounded on $\boldsymbol{R}$ is a (periodic) **controllable and observable trajectory**. Hence if $(i_1^*(t), i_2^*(t), e^*(t), u^*(t))$ is such a trajectory, and if we consider $(i_1^*(t), i_2^*(t), e^*(t))$ as a reference trajectory, it is expected that we can design a controller such that the actual trajectory $(i_1(t), i_2(t), e(t))$ locally exponentially approaches the reference trajectory $(i_1^*(t), i_2^*(t), e^*(t))$. From now on, we examine it.

For simplicity, we suppose that $L_1 = L_2 = c = I_0 = k = 1$. Then (4.62) is equivalent to

$$\begin{cases} \frac{di_1}{dt} = -e + u, \\ \frac{di_2}{dt} = e, \\ 0 = e + (\exp(e) - 1) + i_2 - i_1, \\ y = i_1, \end{cases} . \tag{4.63}$$

To design a controller such that the actual trajectory locally exponentially approaches the reference trajectory, we linearize system (4.63) as follows:

$$\frac{d}{dt}\begin{pmatrix} i_{1,\epsilon} \\ i_{2,\epsilon} \end{pmatrix} = \underbrace{\frac{1}{1+\exp(e^*(t))}\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}}_{A(t)}\begin{pmatrix} i_{1,\epsilon} \\ i_{2,\epsilon} \end{pmatrix} + \underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{B} u_\epsilon, \tag{4.64}$$

$$y_\epsilon = \underbrace{\begin{pmatrix} 1 & 0 \end{pmatrix}}_{C}\begin{pmatrix} i_{1,\epsilon} \\ i_{2,\epsilon} \end{pmatrix}, \tag{4.65}$$

where $i_{1,\epsilon} := i_1(t) - i_1^*(t)$, $i_{2,\epsilon}(t) := i_2(t) - i_2^*(t)$, $u_\epsilon(t) := u(t) - u^*(t)$.

### 4.3.1 Feedback controller design based on LQ optimal control

In order to design a controller such that the actual trajectory locally exponentially approaches the reference trajectory, let us design a feedback controller based on LQ optimal control explained in subsection 2.4.2. Let a reference trajectory of system (4.63) be

$$\begin{cases} i_1^*(t) = \sin(t) + \cos(t) + \exp(\cos(t)) - 1, \\ i_2^*(t) = \sin(t), \\ e^*(t) = \cos(t). \end{cases} \tag{4.66}$$

As shown in example 3.2, since system (4.63) is differentially flat with a flat output $i_2$, we can design a feedforward control. In fact, by relation (3.24), an appropriate feedforward control $u^*(t)$ is derived as

$$u^*(t) = 2\cos(t) - \sin(t)(1 + \exp(\cos(t))). \tag{4.67}$$

We note that the trajectory $(i_1^*(t), i_2^*(t), e^*(t), u^*(t))$ is a **periodic controllable and observable** trajectory. We put $R(t)$ in (2.59) as $R(t) = 5$. Then since linearized system (4.64)-(4.65) is completely controllable and completely observable, the Riccati equation (2.61) has the unique positive definite periodic solution [3,4].

If we apply $u_\epsilon = -\frac{1}{5}B^T P(t)\begin{pmatrix} i_{1,\epsilon} \\ i_{2,\epsilon}\end{pmatrix}$ into (4.64)-(4.65), the origin of the resulting closed-loop is exponentially stable [3,4]. Then as mentioned in section 2.1, if we apply a feedforward and feedback control

$$u(t) = u^*(t) + \left(-\frac{1}{5}B^T P(t)\begin{pmatrix} i_1(t) - i_1^*(t) \\ i_2(t) - i_2^*(t)\end{pmatrix}\right) \tag{4.68}$$

into system (4.63), the actual trajectory $(i_1(t), i_2(t), e(t))$ locally exponentially approaches the reference trajectory $(i_1^*(t), i_2^*(t), e^*(t))$. To see this, applying (4.68) into system (4.63), we simulate the actual trajectory $(i_1(t), i_2(t), e(t))$ of the resulting closed-loop. Fig. 4.16 illustrates the behavior of the resulting closed-loop by applying under $(i_1(0), i_2(0)) = (\exp(1) + 1.5, -1)$ although $(i_1^*(0), i_2^*(0)) = (\exp(1), 0)$, where $e(0)$ was calculated from the third algebraic equation in (4.76) by the Newton method.

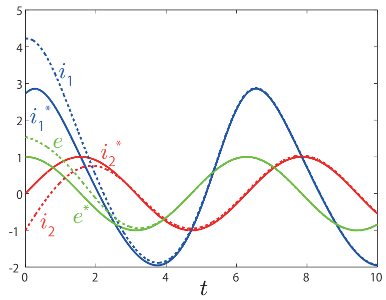



Figure 4.16: The behavior of $(i_1(t), i_2(t), e(t))$.

### 4.3.2 Feedback controller design based on LMI

If linearized system (4.64)-(4.65) has a large period, it is difficult to apply a feedback controller design based on LQ optimal control. However, we can also use a linear matrix inequality (LMI) technique [6] to design a stabilizing feedback controller.

If we apply a feedback control $u_\epsilon = K(t)i_\epsilon$ to system (4.64)-(4.65), then we have the closed-loop

$$\dot{i}_\epsilon = (A(t) + BK(t))i_\epsilon. \tag{4.69}$$

In examples 3.3, we have shown that system (4.63) is uniformly completely controllable if $(i_1^*(t), i_2^*(t), e^*(t), u^*(t))$ is a periodic controllable trajectory. Hence as mentioned in section 2.1, we can design a feedback gain $K(t)$ such that $i_\epsilon = 0$ of (4.69) is exponentially stable. For simplicity, let us consider to find a constant gain $K$ such that $i_\epsilon = 0$ of (4.69) is exponentially stable instead of a time varying gain $K(t)$

Let $P \in \boldsymbol{R}^{2\times 2}$ be a positive definite symmetric matrix and to analyze stability of the origin of system (4.69), we introduce

$$V(i_\epsilon) := i_\epsilon^T P i_\epsilon. \tag{4.70}$$

The derivative of $V(i_\epsilon)$ along the trajectories of (4.69) is given by

$$\dot{V}\big|_{(4.69)}(t, i_\epsilon) := \frac{\partial V}{\partial i_\epsilon}(A(t) + BK)i_\epsilon = 2i_\epsilon^T P(A(t) + BK)i_\epsilon.$$

If there exists an $r > 0$ such that

$$\dot{V}\big|_{(4.69)}(t, i_\epsilon) \leq -2rV(i_\epsilon) \tag{4.71}$$

for all $i_\epsilon \in \boldsymbol{R}^2$ and for all $t \geq 0$, the origin of system (4.69) is exponentially stable [48].

If $A(t)$ can be expressed by $A(t) = A + \Delta_A(t)$, where $||\Delta_A(t)|| \leq \delta$ for all $t \geq 0$, we can apply an LMI-based controller design [6]. In fact, then we have

$$\begin{aligned}\dot{V}\big|_{(4.69)}(t, i_\epsilon) &= i_\epsilon^T(P(A + BK) + (A + BK)^T P)i_\epsilon + 2i_\epsilon^T P\Delta_A i_\epsilon \\ &\leq i_\epsilon^T(P(A + BK) + (A + BK)^T P + P\Delta_A + \Delta_A^T P)i_\epsilon \\ &= \tilde{i}_\epsilon^T(A\tilde{P} + BY + \tilde{P}A^T + Y^T B^T + \Delta_A\tilde{P} + \tilde{P}\Delta_A^T)\tilde{i}_\epsilon,\end{aligned} \tag{4.72}$$

where $\tilde{P} := P^{-1}$, $i_\epsilon = \tilde{P}\tilde{i}_\epsilon$, $Y := K\tilde{P}$. Now suppose that

$$\tilde{P} < \rho I. \tag{4.73}$$

Then since

$$\tilde{i}_\epsilon^T(\Delta_A\tilde{P} + \tilde{P}\Delta_A^T)\tilde{i}_\epsilon = 2\tilde{i}_\epsilon^T\Delta_A\tilde{P}\tilde{i}_\epsilon \leq 2\delta\rho\tilde{i}_\epsilon^T\tilde{i}_\epsilon,$$

(4.72) implies that

$$\dot{V}\big|_{(4.69)}(t, i_\epsilon) \leq \tilde{i}_\epsilon^T(A\tilde{P} + BY + \tilde{P}A^T + Y^T B^T + 2\delta\rho I)\tilde{i}_\epsilon.$$

Hence if there exists $r > 0$ such that

$$A\tilde{P} + BY + \tilde{P}A^T + Y^T B^T + 2\delta\rho I \leq -2r\tilde{P}, \tag{4.74}$$

then (4.71) holds. In summary, we have the following theorem.

**Theorem 4.3** *Suppose that $A(t)$ can be expressed by $A(t) = A + \Delta_A(t)$ and $r > 0$ is given, where $||\Delta_A(t)|| \leq \delta$ for all $t \geq 0$. If there exist $0 < \tilde{P} \in \boldsymbol{R}^{2\times 2}$, $Y \in \boldsymbol{R}^{1\times 2}$, $\rho > 0$ such that (4.73) and (4.74) are satisfied, then the feedback gain $K := Y\tilde{P}^{-1}$ exponentially stabilizes the origin of system (4.69).*

## Simulation: State feedback

By solving LMIs (4.73)-(4.74), we can obtain a feedback gain $K$ such that the origin of closed loop (4.69) is exponentially stable. Then as mentioned in section 2.1, if we apply a feedforward and feedback control

$$u(t) = u^*(t) + Ki_\epsilon \tag{4.75}$$

into system (4.69), the actual trajectory $(i_{1,\epsilon}(t), i_{2,\epsilon}(t), e(t), u(t))$ locally exponentially approaches the reference trajectory $(i^*_{1,\epsilon}(t), i^*_{2,\epsilon}(t), e^*(t), u^*(t))$. To see this, applying (4.75) into system (4.63), we simulate the actual trajectory $(x(t), y(t), \theta(t))$ of the resulting closed-loop. In this case, we should simulate

$$\begin{cases} \frac{di_1}{dt} = -e + (u^*(t) + K(i(t) - i^*(t))), \\ \frac{di_2}{dt} = e, \\ 0 = e + (\exp(e) - 1) + i_2 - i_1 \end{cases} . \tag{4.76}$$

Let a reference trajectory of system (4.63) be (4.66). Since $A(t)$ in (4.64) can be expressed by

$$A(t) = \frac{1}{2}\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}, \quad ||\Delta_A(t)|| \leq \frac{1}{2} \quad \text{for all} \quad t \geq 0,$$

we can apply theorem 4.3. In fact, under $r = 0.1$, we got a feedback gain

$$K = \begin{pmatrix} -8.1130 & -4.4997 \end{pmatrix}. \tag{4.77}$$

Fig. 4.17 illustrates the behavior of closed-loop (4.62) by using the feedback gain (4.77) under $(i_1(0), i_2(0)) = (\exp(1) + 1.5, -1)$ although $(i_1^*(0), i_2^*(0)) = (\exp(1), 0)$, where $e(0)$ was calculated from the third algebraic equation in (4.76) by the Newton method.

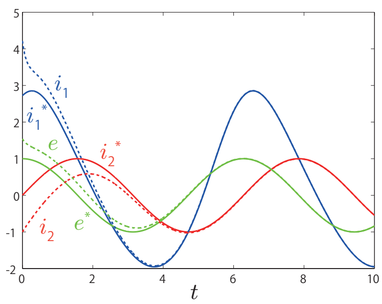



Figure 4.17: The behavior of $(i_1(t), i_2(t), e(t))$.

Moreover, Fig. 4.18 illustrates feedback signals of the LQ optimal control in subsection 4.3.1 and the LMI method. We can see that the LMI method uses large signals compared to the LQ optimal control method.

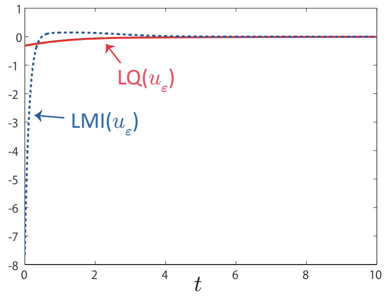



Figure 4.18: Feedback signals of LQ optimal control and LMI methods.

## Simulation: State-estimate feedback

If the available signal in system (4.76) is only output signal $y$, we cannot use the control (4.75). Alternatively we need to design an appropriate state observer. In this case, to see exponential stability of the reference trajectory $(i_1^*(t), i_2^*(t), e^*(t))$, we should simulate

$$\begin{cases} \frac{di_1}{dt} = -e + u^*(t) + K\hat{i}_\epsilon, \\ \frac{di_2}{dt} = e, \\ 0 = e + \exp(e) - 1 + i_2 - i_1, \\ y_\epsilon = i_1 - i_1^*(t), \\ \frac{d\hat{i}_\epsilon}{dt} = (A(t) + BK)\hat{i}_\epsilon + L(t)(y_\epsilon - \hat{i}_{1,\epsilon}), \end{cases} \tag{4.78}$$

where $L(t)$ is defined by (4.3).

Let a reference trajectory of system (4.62) be (4.66). By relation (3.24), an appropriate feedforward control $u^*(t)$ is given by (4.67). In this case, we have a feedback gain (4.77). Let $T = 5$, $\gamma = 50$, $p = 15$. Fig. 4.19 illustrates the behavior of closed-loop (4.78) by using the feedback gain (4.77) under $(i_1(0), i_2(0)) = (\exp(1) + 1.5, -1)$ although $(i_1^*(0), i_2^*(0)) = (\exp(1), 0)$, where $e(0)$ was calculated from the third algebraic equation in (4.76) by the Newton method.

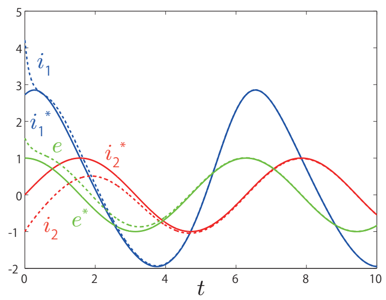



Figure 4.19: The behavior of $(i_1(t), i_2(t), e(t))$.

## 4.4 Summary

We have elaborated the difference between variational and flatness-based trajectory generation methods. Moreover, using nonholonomic mobile robot and simple circuit examples, we have demonstrated that LQ optimal control and LMI methods are useful to design for a trajectory tracking control of algebraically controllable and observable systems. The results of simulations have shown that large initial errors are allowed for the proposed control strategies. An estimation of the domain of attraction is remained for future works.

# Chapter 5

# Conclusion

This thesis has given a class of nonlinear systems and reference trajectories such that trajectory tracking controls are easily realized. We summarize the contributions of the thesis.

- In chapter 2, we have shown that if a given nonlinear system is algebraically controllable (observable), every linearized system along any periodic controllable (observable) trajectory is uniformly completely controllable (observable). Moreover, we have explained that if a given system is algebraically controllable and observable, a linear quadratic optimal control method is useful to design a feedback controller such that the actual trajectory asymptotically approaches the reference trajectory. Furthermore, we have proven that the concepts of algebraic controllability and accessibility are equivalent, and for nonlinear mechanical control systems, we have provided a reduction condition for examining whether or not the system is algebraically controllable.

- In chapter 3, we have also introduced algebraic controllability and algebraic observability of nonlinear differential algebraic systems (DAS) with geometric index one. We have shown that if a given nonlinear DAS with geometric index one is algebraically controllable (observable), every linearized system along any periodic controllable (observable) trajectory is uniformly completely controllable (observable). Moreover, we have given the definition of differential flatness of DAS which does not distinguish state, input, and output variables, and provided how to produce other flat outputs from one flat output.

- In chapter 4, we have clarified the difference between variational and flatness-based trajectory generation methods. Moreover, using a nonholonomic mobile robot and a simple circuit model, we have demonstrated that trajectory tracking controls of algebraically controllable and algebraically observable systems are easily realized. The results of simulations have shown that large initial errors are allowed for the proposed control strategies.

The following questions are open problems.

- The key concepts of controllable trajectory and observable trajectory revolve about piecewise smooth functions. How can we extend the concepts of the trajectories?

- Simulation results in chapter 4 have shown that large initial errors are allowed for the proposed control strategy which is composed of a feedforward control and a linear feedback control. This is valuable for practical applications because a linear feedback controller is simple compared with nonlinear feedback controllers. Thus it is important to examine how initial errors are allowed. In order to investigate it, we have to study the domain of attraction of a closed-loop nonlinear time varying error system such as (2.7). Although Lyapunov function approaches can be applied to an estimation of the domain of attraction [12, 48], in general, it is difficult to construct a Lyapunov function for a nonlinear time varying system. Hence it is also desirable to develop another approach. For example, how can we expand numerical analysis approaches for time invariant systems based on references [92–94, 106, 107] into time varying systems?

# Appendix A

# Algebra

For the convenience of readers, we summarize some results of algebra. In particular, the contents of the appendix are applied in appendix B. We refer to [15,37,38].

Let $G$ be a set together with a binary operation $\cdot : G \times G \to G$. The set $G$ is called a **semi-group** if

$$(a \cdot b) \cdot c = a \cdot (b \cdot c)$$

for all $a, b, c \in G$. In addition, if the semi-group $G$ has the **identity element**, that is, there exists an element $e \in G$ such that

$$a \cdot e = e \cdot a = a$$

for all $a \in G$, $G$ is called a **monoid**. Furthermore, for each $a \in G$, if the monoid $G$ has the **inverse element**, that is, there exists $b \in G$ such that

$$a \cdot b = b \cdot a = e,$$

the monoid $G$ is called a **group**. Moreover, if the group $G$ satisfies

$$a \cdot b = b \cdot a,$$

for all $a, b \in G$, the group $G$ is called an **Abelian group**.

Let $R$ be a set together with two binary operations $+ : R \times R \to R$ and $\cdot : R \times R \to R$. The set $R$ is called a **ring** if $R$ is an Abelian group and a monoid under $+$ and $\cdot$, respectively, and for all $a, b, c \in R$,

$$a \cdot (b + c) = a \cdot b + a \cdot c,$$
$$(b + c) \cdot a = b \cdot a + c \cdot a.$$

The ring $R$ is called a **domain** if for all $r_1$, $r_2 \in R$,

$$r_1 r_2 = 0 \Rightarrow r_1 = 0 \quad \text{or} \quad r_2 = 0.$$

Furthermore, if the ring $R$ satisfies

$$a \cdot b = b \cdot a$$

for all $a, b \in R$, the ring $R$ is called a **commutative ring**.

Let $F$ be a ring together with two binary operations $+ : R \times R \to R$ and $\cdot : R \times R \to R$. The ring $F$ is called a **skew field** if $F$ has the inverse element for all $a \in F \backslash \{0\}$. The set $F$ is called a **field** if $F$ is a commutative ring and a skew field.

Let $R$ be a ring. An abelian group $M$ together with an operation $R \times M \to M$ is called a left $R$-**module** if for all $a, b \in R$ and for all $x, y \in M$,

1. $1_R \cdot x = x$,
2. $a(x + y) = ax + ay$,
3. $(a + b)x = ax + bx$,
4. $(ab)x = a(bx)$.

Similarly, a right $R$-module is defined. An element $m \in M$ is called **torsion** if there exists $0 \neq r \in R$ such that $rm = 0$. The module $M$ is called **torsion** if any element in $M$ is torsion. The module $M$ is called **torsion-free** if it has no torsion elements except zero, that is, for all $r \in R$, $m \in M$, we have

$$rm = 0 \Rightarrow r = 0 \text{ or } m = 0.$$

Let $M$ be a left $R$-module and $X \subset M$. The set $X$ is called **system of generators** of $M$ if $M = \sum_{x \in X} Rx$. In particular, if we can take $X$ from a finite set, $M$ is called **finitely generated**. A subset $X \subset M$ is called $R$-**linearly independent** if all $m_1, \cdots, m_k \in X$, $k \geq 1$ satisfy

$$r_1 m_1 + \cdots + r_k m_k = 0 \Leftrightarrow r_1 = \cdots = r_k = 0$$

A subset $X \subset M$ is called a **basis** of $M$ if

1. $X$ is a system of generators,
2. $X$ is $R$-linearly independent.

A left $R$-module $M$ is called **free** if $M$ has a basis. Consider $\{M_i \mid i \in \mathcal{I}\}$, where $M_i$ are left $R$-modules, and $M := \Pi_{i \in \mathcal{I}} M_i$. For any $(x_i)_{i \in \mathcal{I}}$, $(y_i)_{i \in \mathcal{I}} \in M$ and $a \in R$, we define

1. $(x_i)_{i \in \mathcal{I}} + (y_i)_{i \in \mathcal{I}} = (x_i + y_i)_{i \in \mathcal{I}}$,
2. $a \cdot (x_i)_{i \in \mathcal{I}} = (ax_i)_{i \in \mathcal{I}}$.

Then $M$ becomes a left $R$-module and $M$ is called a **direct product** of $\{M_i \,|\, i \in \mathcal{I}\}$.

Let $R$ be a ring and $M$, $N$ left $R$-modules. A map $\phi : M \to N$ is called an $R$-**homomorphism** if for any $x, y \in M$ and $a \in R$,

1. $\phi(x + y) = \phi(x) + \phi(y)$,
2. $\phi(ax) = a\phi(x)$.

Let

$$\mathrm{Hom}_R(M, N) := \{\phi : M \to N \,|\, \phi \,\text{is}\, R\text{-homomorphism}\}\,.$$

For $\phi, \psi \in \mathrm{Hom}_R(M, N)$, we define $\phi + \psi : M \to N$ as

$$(\phi + \psi)(x) := \phi(x) + \psi(x).$$

Then $\phi+\psi \in \mathrm{Hom}_R(M, N)$. Furthermore, we define $-\phi : M \to N$ as $(-\phi)(x) := -(\phi(x))$. Then $\mathrm{Hom}_R(M, N)$ becomes an Abelian group. An $R$-homomorphism $\phi : M \to N$ is called an $R$-**isomorphism** if $\phi$ is injective and surjective. If there exists an $R$-isomorphism from $M$ to $N$, we write $M \cong N$.

Let $M$ be a left $R$-module and $L \subset M$. The set $L$ is called a left $R$-**submodule** of $M$ if

1. $x, y \in L \Rightarrow x + y \in L$,
2. $a \in R$, $x \in L \Rightarrow ax \in L$.

Similarly, a right $R$-submodule is defined. Consider $\{M_i \,|\, i \in \mathcal{I}\}$, where $M_i$ are left $R$-modules, and define

$$\bigoplus_{i\in\mathcal{I}} M_i := \left\{(x_i)_{i\in\mathcal{I}} \in \Pi_{i\in\mathcal{I}} M_i \,\middle|\, x_i = 0 \text{ except for finite number of } x_j\right\}.$$

The set $\bigoplus_{i\in\mathcal{I}} M_i$ is a left $R$-submodule of $\Pi_{i\in\mathcal{I}} M_i$ and is called a **direct sum**. We note that $R$ is a left and right $R$-module. A left $R$-submodule of $R$ is called a left **ideal**. Similarly, a right ideal is defined. For an $R$-homomorphism $\phi : M \to N$,

$$\mathrm{Ker}(\phi) := \{x \in M \,|\, \phi(x) = 0\}\,,$$
$$\mathrm{Im}(\phi) := \{\phi(x) \,|\, x \in M\}$$

are submodules of $M$ and $N$, respectively. We note that any left $R$-module $M$ has submodules $\{0\}$ and $M$. These submodules are called **trivial** submodules. A non-zero left $R$-module $M$ is called **simple** if all submodules of $M$ are trivial submodules. A left $R$-module $M$ is called a left **Noetherian module** if the following equivalent conditions are satisfied:

1. Every ascending chain $M_0 \subset M_1 \subset \cdots$ of left submodules in $M$ must become stationary.
2. Every left submodule $N$ in $M$ is finitely generated.
3. Every non-empty family of left submodules in $M$ has a maximal element.

A ring $R$ is called a left **Noetherian ring** if the ring is a left Noetherian module as a left module.

We say that a ring $R$ has the **left Ore property** [85, 86] if for any $0 \neq r_1, r_2 \in R$, there exist $0 \neq r_1', r_2' \in R$ such that $r_1' r_1 = r_2' r_2$. Similarly, the right Ore property is defined. We note that if $R$ is a commutative ring, $R$ has the left and right Ore property.

The following propositions are used in appendix B.

**Proposition A.1** *A finitely generated left module over a left Noetherian ring is a Noetherian module.*

**Proposition A.2** *If $R$ is a left Noetherian domain, then it has the left Ore property.*

**Proof** Let $0 \neq r_1, r_2 \in R$. Consider the left ideals

$$\mathcal{I}_n := \sum_{i=0}^{n} R r_1 r_2^i.$$

Then we have an ascending chain $\mathcal{I}_0 \subset \mathcal{I}_1 \subset \cdots$, which must become stationary by the Noetherian property. Let $n$ be the smallest integer such that $\mathcal{I}_{n+1} = \mathcal{I}_n$. Then

$$a_{n+1} r_1 r_2^{n+1} = \sum_{i=0}^{n} a_i r_1 r_2^i$$

for some $a_i \in R$. Re-arranging the summands, we obtain

$$a_0 r_1 = \left( a_{n+1} r_1 r_2^n - \sum_{i=1}^{n} a_i r_1 r_2^{i-1} \right) r_2$$

and hence we have constructed a left common multiple. If the coefficients were zero, we have

$$a_{n+1} r_1 r_2^n = \sum_{i=0}^{n-1} r_1 r_2.$$

Hence $\mathcal{I}_n = \mathcal{I}_{n-1}$, contradicting the minimality of $n$. $\square$

**Proposition A.3** *A domain $R$ admits a field of left fractions*

$$K = \left\{ r^{-1}s \,|\, r, s \in R,\, r \neq 0 \right\}$$

*if and only if $R$ has the left Ore property.*

**Proof** Suppose that $R$ admits a field of left fractions $K$. Then for any $r$, $s \in R$, $r \neq 0$, we have

$$\begin{cases} s = 1^{-1}s, \\ r^{-1} = r^{-1}1. \end{cases}$$

Thus $r^{-1}$, $s \in K$. Since $K$ is a field, $sr^{-1} \in K$. Hence there exist $r_1$, $s_1 \in R$, $r_1 \neq 0$ such that $sr^{-1} = r_1^{-1}s_1$. Thus

$$r_1 s = s_1 r.$$

Therefore all $r$, $s \in R$, $r \neq 0$ have a left common multiple. In addition, since $R$ is a domain, $n \neq 0$ implies that $n_1 \neq 0$. Thus $R$ has the left Ore property.

Conversely, let $R$ be a left Ore domain, and $R^* := R \backslash \{0\}$. We define a relation on $R^* \times R$ by

$$(r_1, s_1) \sim (r_2, s_2) :\Leftrightarrow \text{ for some } c_1, c_2 \in R^*,\ c_1 r_1 = c_2 r_2 \text{ implies } c_1 s_1 = c_2 s_2.$$

This is an equivalence relation. Let

$$K := (R^* \times R)/\sim = \{[(r, s)] \,|\, (r_1, s_1) \sim (r_2, s_2) \text{ for all } (r_1, s_1),\, (r_2, s_2) \in [(r, s)]\}.$$

We define the multiplication on $K$ by

$$[(r_1, s_1)] \cdot [(r_2, s_2)] := [(ar_1, bs_2)],$$

where $as_1 = br_2$, $a \neq 0$. This is well-defined. Let $0_K := [(1, 0)] = [(r, 0)]$ for all $r \neq 0$, and $1_K := [(1, 1)] = [(r, r)]$ for all $r \neq 0$. Then for all $k \in K$,

$$\begin{cases} 0_K \cdot k = k \cdot 0_K = 0_K, \\ 1_K \cdot k = k \cdot 1_K = k. \end{cases}$$

For all $0_K \neq [(r, s)] \in K$, there exists an inverse element. In fact,

$$\begin{cases} [(r, s)] \cdot [(s, r)] = [(ar, br)] = [(ar, ar)] = 1_K, \\ [(s, r)] \cdot [(r, s)] = [(as, bs)] = [(as, as)] = 1_K. \end{cases}$$

To define the addition on $K$, it suffices to define $k + 1_K$ for all $k \in K$ because the addition of all $k, l \in K$ can be defined by

$$k + l := \begin{cases} k & (l = 0_K), \\ l(l^{-1}k + 1_K) & (l \neq 0_K). \end{cases}$$

So, we set

$$k + 1_K = [(r, s)] + [(1, 1)] := [(r, s + r)].$$

Hence $K$ becomes a field, and we have an injective ring homomorphism

$$R \to K, \quad r \mapsto [(1, r)].$$

Identifying $R$ with its image under this map, we have for all $r \neq 0$,

$$r^{-1}s = [(1, r)]^{-1} \cdot [(1, s)] = [(r, 1)] \cdot [(1, s)] = [(r, s)].$$

Therefore an element of $K$ as constructed can be identified with a left fraction of elements of $R$. $\square$

Let $M$ be a left $R$-module and $N$ a submodule of $M$. We define $\sim$ as follows.

$$\text{For all}\, x, y \in M, \quad x \sim y :\Leftrightarrow x - y \in N.$$

The relation $\sim$ is an equivalence relation on $M$ and the equivalence class of $x \in M$ is expressed by

$$x + N := \{x + z \,|\, z \in N\}.$$

Moreover, we define

$$M/N := \{x + N \,|\, x \in M\}$$

and for all $x + N, y + N \in M/N$ and $a \in R$,

1. $(x + N) + (y + N) := (x + y) + N$,
2. $a(x + N) := ax + N$.

Then $M/N$ becomes a left $R$-module and the module is called a **quotient module**.

# Appendix B

# Algebraic linear system theory

For the convenience of readers, we summarize some results of algebraic linear system theory based on reference [117, 118]. Let $\mathcal{D}$ be a ring and let $\mathcal{F}$ be a left $\mathcal{D}$-module. Let

$$\mathcal{B} := \{w \in \mathcal{F}^q \mid Rw = 0\}$$

be a behavior [111–113], where $R \in \mathcal{D}^{g\times q}$. Let the **system module**

$$\mathcal{M} := \mathcal{D}^{1\times q} / \left(\mathcal{D}^{1\times g} R\right).$$

According to the **Malgrange isomorphism** [71], the group isomorphism

$$\mathcal{B} \cong \mathrm{Hom}_{\mathcal{D}}(\mathcal{M}, \mathcal{F}), \quad w \mapsto \phi_w$$

holds, where $\phi_w : \mathcal{M} \to \mathcal{F}$, $x + M \mapsto xw$ for all $x \in \mathcal{D}^{1\times q}$. We note that the Malgrange isomorphism relates the analytic object $\mathcal{B}$ and the algebraic object $\mathcal{M}$. Let $\mathcal{M}_1$, $\mathcal{M}_2$, $\mathcal{M}_3$ be left $\mathcal{D}$-modules and let $f_1 : \mathcal{M}_1 \to \mathcal{M}_2$ and $f_2 : \mathcal{M}_2 \to \mathcal{M}_3$ be $\mathcal{D}$-homomorphisms. A sequence

$$\mathcal{M}_1 \xrightarrow{f_1} \mathcal{M}_2 \xrightarrow{f_2} \mathcal{M}_3 \tag{B.1}$$

is called **exact** if $\mathrm{Im} f_1 = \mathrm{Ker} f_2$. A left $\mathcal{D}$-module $\mathcal{F}$ is called **injective** if $\mathrm{Hom}_{\mathcal{D}}(\cdot, \mathcal{F})$ is an exact contravariant functor, that is, for left $\mathcal{D}$-modules $\mathcal{M}_1$, $\mathcal{M}_2$, $\mathcal{M}_3$, if (B.1) is exact, then

$$\mathrm{Hom}_{\mathcal{D}}(\mathcal{M}_1, \mathcal{F}) \xleftarrow{\mathrm{Hom}_{\mathcal{D}}(f_1,\mathcal{F})} \mathrm{Hom}_{\mathcal{D}}(\mathcal{M}_2, \mathcal{F}) \xleftarrow{\mathrm{Hom}_{\mathcal{D}}(f_2,\mathcal{F})} \mathrm{Hom}_{\mathcal{D}}(\mathcal{M}_3, \mathcal{F}), \tag{B.2}$$

is also exact, where

$$\mathrm{Hom}_{\mathcal{D}}(f, \mathcal{F}) : \mathrm{Hom}_{\mathcal{D}}(\mathcal{M}_2, \mathcal{F}) \to \mathrm{Hom}_{\mathcal{D}}(\mathcal{M}_1, \mathcal{F}), \quad \phi \mapsto \phi \circ f.$$

Furthermore, a left $\mathcal{D}$-module $\mathcal{F}$ is called **injective cogenerator** if exactness of (B.1) and (B.2) are equivalent.

# B.1 Autonomy

In this section, we assume that the left $\mathcal{D}$-module $\mathcal{F}$ is injective cogenerator. Let us consider the projection of the behavior $\mathcal{B}$ onto the $i$-th component

$$\pi_i : \mathcal{B} \to \mathcal{F}, \quad w \mapsto w_i.$$

The variable $w_i$ is called **free variable** of $\mathcal{B}$ if $\pi_i$ is surjective. The behavior $\mathcal{B}$ is called **autonomous** if it admits no free variables.

**Lemma B.1** *If $\mathcal{M}$ is torsion, then $\mathcal{B}$ is autonomous.*

**Proof** If $\mathcal{B}$ is not autonomous, then there exists an exact sequence

$$\mathcal{B} \xrightarrow{\pi_i} \mathcal{F} \to 0.$$

By the Malgrange isomorphism, the exact sequence is equivalent to an exact sequence

$$\mathrm{Hom}_{\mathcal{D}}(\mathcal{M}, \mathcal{F}) \to \mathrm{Hom}_{\mathcal{D}}(\mathcal{D}, \mathcal{F}) \to 0.$$

Since $\mathcal{F}$ is injective cogenerator,

$$\mathcal{M} \xleftarrow{i} \mathcal{D} \leftarrow 0$$

is also exact. Hence $i$ is injective. Let $m := i(1) \neq 0$. If $dm = 0$, then $di(1) = i(d) = 0$. Thus since $i$ is injective, $d = 0$. Hence $m$ is not a torsion element. Therefore $\mathcal{M}$ is not torsion. □

In order to get the converse direction of the implication of the above lemma, we assume that $\mathcal{D}$ is a **left Noetherian domain**. If $\mathcal{D}$ is left Noetherian, the finitely generated $\mathcal{D}$-module is a left Noetherian module (see proposition A.1 in appendix A). Then by lemma B.1 and proposition A.2, we have the following proposition.

**Proposition B.1** *The module $\mathcal{M}$ is torsion if and only if $\mathcal{B}$ is autonomous.*

**Proof** From lemma B.1, if $\mathcal{M}$ is torsion, $\mathcal{B}$ is autonomous. Thus it suffices to show the converse. Assume that $\mathcal{M}$ is not torsion. We first show that there exists an integer $1 \leq i \leq q$ such that $[e_i]$ is not torsion, where $e_i$ denotes the $i$-th natural basis vector of $\mathcal{D}^{1\times q}$, and where $[e_i]$ denotes the residue class of $e_i$ modulo $\mathcal{D}^{1\times g}R$. Suppose that all $[e_i]$ were torsion, that is, $d_i[e_i] = 0$ for some $d_i \neq 0$. Now let $m \in \mathcal{M}$ be given. Then $m = [x]$ for some $x \in \mathcal{D}^{1\times q}$. Hence

$$m = [x] = [\sum_{i=1}^{q} x_i e_i] = \sum_{i=1}^{q} x_i e_i + \mathcal{D}^{1\times g}R = \sum_{i=1}^{q} x_i(e_i + \mathcal{D}^{1\times g}R) = \sum_{i=1}^{q} x_i[e_i],$$

where $x_i \in \mathcal{D}$. Since $\mathcal{D}$ is a left Noetherian domain, by proposition A.2, $\mathcal{D}$ has the left Ore property. By the left Ore property, there exist $0 \neq b_i, c_i \in \mathcal{D}$ such that $b_i d_i = c_i x_i$. Similarly, for $c_i$, there exist $0 \neq a_i$ such that $a := a_i c_i$, $1 \leq i \leq q$. Then

$$am = \sum_{i=1}^{q} a x_i [e_i] = \sum_{i=1}^{q} a_i c_i x_i [e_i] = \sum_{i=1}^{q} a_i b_i d_i [e_i] = 0.$$

Hence $\mathcal{M}$ is torsion, contracting the assumption.

Let $f : \mathcal{D} \to \mathcal{M}$ be a $\mathcal{D}$-homomorphism and let $[e_i]$ be not torsion. Let $f(1) := [e_i]$. Since $[e_i]$ is not torsion, for any $0 \neq d \in \mathcal{D}$, we have $f(d) = df(1) = d[e_i] \neq 0$. Hence $f$ is injective. Thus there exists an exact sequence

$$0 \to \mathcal{D} \xrightarrow{f} \mathcal{M}.$$

Since $\mathcal{F}$ is injective,

$$0 \leftarrow \mathrm{Hom}_{\mathcal{D}}(\mathcal{D}, \mathcal{F}) \leftarrow \mathrm{Hom}_{\mathcal{D}}(\mathcal{M}, \mathcal{F}).$$

The Malgrange isomorphism implies that

$$0 \leftarrow \mathcal{F} \xleftarrow{p} \mathcal{B}$$

is also exact. Hence $p$ is surjective. Furthermore we can show that $p \equiv \pi_i$. Therefore $\mathcal{B}$ is not autonomous. □

## B.2 Image representation

In this section, we assume that $\mathcal{D}$ is a left Noetherian domain and the left $\mathcal{D}$-module $\mathcal{F}$ is injective cogenerator. We say that the behavior $\mathcal{B}$ admits an **image representation** if there exists $L \in \mathcal{D}^{q \times p}$ such that

$$\mathcal{B} = \{w \in \mathcal{F}^q \mid \exists l \in \mathcal{F}^p \text{ s.t. } w = Ll\}.$$

**Lemma B.2** *The behavior $\mathcal{B}$ admits an image representation if and only if $R$ is a* **left syzygy matrix***, that is, there exists a $\mathcal{D}$-matrix $L$ such that* $\mathrm{Im}_{\mathcal{D}}(\cdot R) = \mathrm{Ker}_{\mathcal{D}}(\cdot L)$.

**Proof** The behavior $\mathcal{B}$ admits an image representation if and only if $\mathrm{Ker}_{\mathcal{F}}(R) = \mathrm{Im}_{\mathcal{F}}(L)$. Hence there exists an exact sequence

$$\mathcal{F}^p \xrightarrow{L} \mathcal{F}^q \xrightarrow{R} 0.$$

By the Malgrange isomorphism,

$$\mathrm{Hom}_{\mathcal{D}}(\mathcal{D}^{1\times p},\mathcal{F}) \xrightarrow{\mathrm{Hom}_{\mathcal{D}}(\cdot L,\mathcal{F})} \mathrm{Hom}_{\mathcal{D}}(\mathcal{D}^{1\times q},\mathcal{F}) \xrightarrow{\mathrm{Hom}_{\mathcal{D}}(\cdot R,\mathcal{F})} 0$$

is also exact. Since $\mathcal{F}$ is injective cogenerator,

$$\mathcal{D}^{1\times p} \overset{\cdot L}{\longleftarrow} \mathcal{D}^{1\times g} \overset{\cdot R}{\longleftarrow} 0$$

is also exact. Therefore

$$\mathrm{Im}_{\mathcal{D}}(\cdot R) = \mathrm{Ker}_{\mathcal{D}}(\cdot L).$$

□

**Lemma B.3** *If the behavior $\mathcal{B}$ admits an image representation, then $\mathcal{M}$ is torsion-free.*

**Proof** Let $0 \neq d \in \mathcal{D}$ and $x \in \mathcal{D}^{1\times q}$ be such that $dx \in \mathrm{Im}(\cdot R)$. Since by lemma B.2, there exists a $\mathcal{D}$ matrix $L$ such that $\mathrm{Im}_{\mathcal{D}}(\cdot R) = \mathrm{Ker}_{\mathcal{D}}(\cdot L)$, we have $dxL = 0$. Since $\mathcal{D}$ is a domain, $xL = 0$. Hence $x \in \mathrm{Ker}_{\mathcal{D}}(\cdot L) = \mathrm{Im}_{\mathcal{D}}(\cdot R)$. □

In order to get the converse direction of lemma B.3, let the domain $\mathcal{D}$ be **Noetherian**, that is, both left and right Noetherian. Then we have the following proposition.

**Proposition B.2** *The following are equivalent:*

1. *$\mathcal{B}$ admits an image representation.*
2. *$\mathcal{M}$ is torsion-free.*
3. *$R$ is a left syzygy matrix.*

**Proof** By lemma B.2, we have the equivalence of assertions 1 and 3. Furthermore by lemma B.3 the implication "1 ⇒ 2" follows. Thus it suffices to show "2 ⇒ 3". It is known that every finitely generated torsion-free module over a Noetherian domain can be embedded into a finitely generated free module [30]. Hence the exact sequence

$$\mathcal{D}^{1\times g} \xrightarrow{\cdot R} \mathcal{D}^{1\times q} \xrightarrow{\pi} \mathcal{M} = \mathcal{D}^{1\times q}/\mathrm{Im}_{\mathcal{D}}(\cdot R)$$

and the embedding $i : \mathcal{M} \to \mathcal{D}^{1\times p}$ yields an exact sequence

$$\mathcal{D}^{1\times g} \xrightarrow{\cdot R} \mathcal{D}^{1\times q} \xrightarrow{i\circ\pi} \mathcal{D}^{1\times p}$$

and the map $i \circ \pi$ has to take the form $\cdot L$ for some $L \in \mathcal{D}^{q\times p}$. □

## B.3 Controllability of one-dimensional systems

This section algebraic analyzes controllability of one-dimensional systems described by ordinary differential equations with meromorphic coefficients. Let $\mathcal{D}_t := \mathcal{M}_t[\frac{d}{dt}]$, where $\mathcal{M}_t$ denotes the field of meromorphic functions depending on $t$. Every $0 \neq a \in \mathcal{D}_t$ can be uniquely expressed by

$$a = a_n(t)\frac{d^n}{dt^n} + \cdots + a_1(t)\frac{d}{dt} + a_0(t),$$

where $a_i(t) \in \mathcal{M}_t$ and $a_n(t) \neq 0$. For any $a = \sum_{i=1}^n a_i(t)\frac{d^i}{dt^i} \in \mathcal{D}_t$, $\frac{d}{dt}a$ is defined as

$$\frac{d}{dt}a := \sum_{i=0}^n \left( a_i(t)\frac{d^{i+1}}{dt^{i+1}} + \dot{a}_i(t)\frac{d^i}{dt^i} \right).$$

Clearly, $\mathcal{D}_t$ is a domain. Furthermore we can show that $\mathcal{D}_t$ is a left and right Euclidean domain. Here, the domain $\mathcal{D}_t$ is called a **left Euclidean domain** if for $b$, $0 \neq a \in \mathcal{D}_t$, there exist $q$, $r \in \mathcal{D}_t$ such that

$$b = aq + r$$

and $\deg r < \deg q$. Similarly, **right Euclidean domain** is defined. In fact, we have the following proposition.

**Proposition B.3** *The ring $\mathcal{D}_t$ is simple (i.e. the only ideals are $0$ and $\mathcal{D}_t$), and it is a left and right Euclidean domain.*

**Proof** First, we show that $\mathcal{D}_t$ is simple. Let $\mathcal{I}$ be a non-zero left and right ideal in $\mathcal{D}_t$ and let

$$n := \min\{\deg f \mid 0 \neq f \in \mathcal{I}\}.$$

Then $\mathcal{I}$ contains an element $d \in \mathcal{D}_t$ of degree $n$. If $n = 0$, then $1 \in \mathcal{I}$, that is, $\mathcal{I} = \mathcal{D}_t$. Let $n \geq 1$. Consider $kd - dk \in \mathcal{I}$, where $k \in \mathcal{M}_t$. Then

$$\begin{aligned} kd - dk &= k\sum_{i=0}^n a_i \frac{d^i}{dt^i} - \sum_{i=0}^n a_i \frac{d^i}{dt^i} k \\ &= k\sum_{i=0}^n a_i \frac{d^i}{dt^i} - \sum_{i=0}^n a_i \sum_{j=0}^i \binom{i}{j} k^{(i-j)} \frac{d^j}{dt^j}. \end{aligned}$$

Furthermore we have

$$\begin{aligned} \sum_{i=0}^n a_i \sum_{j=0}^i \binom{i}{j} k^{(i-j)} \frac{d^j}{dt^j} &= a_0 k + a_1 \binom{1}{0} \dot{k} + \cdots + a_n \binom{n}{0} k^{(n)} \\ &\quad + \left\{ a_1 \binom{1}{1} k + a_2 \binom{2}{1} \dot{k} + \cdots + a_n \binom{n}{1} k^{(n)} \right\} \frac{d}{dt} \\ &\quad + \cdots + a_n k \frac{d^n}{dt^n}. \end{aligned}$$

Hence

$$\begin{aligned}
kd - dk = ka_0 &- \left(a_0 k a_1 \binom{1}{0} \dot{k} + \cdots + a_n \binom{n}{0} k^{(n)}\right) \\
&+ \cdots + \left(ka_{n-1} - a_{n-1}\binom{n-1}{n-1} k - a_n \binom{n}{n-1}\dot{k}\right)\frac{d^{n-1}}{dt^{n-1}} \\
&+ (ka_n - a_n k)\frac{d^n}{dt^n}.
\end{aligned}$$

Since $\mathcal{M}_t$ is commutative, the coefficient $ka_n - a_n k$ at $\frac{d^n}{dt^n}$ equals zero. Thus the degree of $kd-dk$ is at most $n-1$. Since $n$ was chosen to be minimal, we must have $kd - dk = 0$. Then the coefficient at $\frac{d^{n-1}}{dt^{n-1}}$ has to vanish. Hence the coefficient $ka_{n-1} - -a_{n-1}\binom{n-1}{n-1} k - a_n \binom{n}{n-1}\dot{k} = -a_n n\dot{k}$ at $\frac{d^{n-1}}{dt^{n-1}}$ equals zero. Since $a_n, n \neq 0$, we have $\dot{k} = 0$ for all $k \in \mathcal{M}_t$. Since $t \in \mathcal{M}_t$ and $\dot{t} = 1 \neq 0$, this is a contradiction. Therefore $\mathcal{D}_t$ is simple.

Next we show that $\mathcal{D}_t$ is a left and right Euclidean domain. We first observe that for all $a$, $b \in \mathcal{D}_t$, $a \neq 0$, with $\deg b \geq \deg a$, there exists $f \in \mathcal{D}_t$ such that

$$\deg(b - fa) < \deg(b).$$

Indeed if $a = a_n \frac{d^n}{dt^n} + \cdots + a_0$ and $b = b_m \frac{d^m}{dt^m} + \cdots + b_0$ with $a_n$, $b_m \neq 0$ and $n \geq m$, we may take $f = a_n \frac{d^{n-m}}{dt^{n-m}} b_m^{-1}$. Now let $a, b \in \mathcal{D}_t$, $a \neq 0$ be given and let

$$\delta := \min\{\deg(b - fa) \mid f \in \mathcal{D}_t\}.$$

Let $q \in \mathcal{D}_t$ be such that $\deg(b - fa) = \delta$. If $\deg(b - qa) \geq \deg a$, then there exists $f \in \mathcal{D}_t$ such that $\deg(b - qa - fa) < \deg(b - qa) = \delta$. This contradicts the minimality of $\delta$. If $\deg(b - qa) < \deg a$, then putting $r := b - qa$, we have $b = qa + r$ with $\deg r < \deg a$.

The right division with remainder is constructed similarly. □

Hence the ring $\mathcal{D}_t$ is a **left and right principal ideal domain** (i.e. every left ideal and every right ideal can be generated by one single element) [15]. Since a left and right principal ideal domain is a Noetherian domain, by proposition A.2, $\mathcal{D}_t$ has Ore property. Hence $\mathcal{D}_t$ admits a skew field $\mathcal{K}$ of fractions containing elements of the form $k = d^{-1}n$ or $k = nd^{-1}$, where $0 \neq d \in \mathcal{D}_t$ and $n \in \mathcal{D}_t$ [15] (see proposition A.3). Therefore the rank of a matrix $R \in \mathcal{D}_t^{g\times q}$ is well defined via

$$\operatorname{rank} R = \dim\left(\mathcal{K}^{1\times g}R\right) = \dim\left(R\mathcal{K}^q\right).$$

The following proposition [15] is important to characterize controllability.

**Proposition B.4** *Let $R \in \mathcal{D}_t^{g\times q}$. Then there exist unimodular matrices $U \in \mathcal{D}_t^{g\times g}$ and $V \in \mathcal{D}_t^{q\times q}$ such that*

$$URV = \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}, \tag{B.3}$$

*where $D = \mathrm{diag}(1, \cdots, 1, d) \in \mathcal{D}_t^{p\times p}$, $0 \neq d \in \mathcal{D}_t$, and $p := \mathrm{rank}\, R$.*

The form (B.3) is called the **Jacobson form** of $R$ [15, 118]. To give a proof of proposition B.4, we need some preparations. An element $a \in \mathcal{D}_t$ is called a **right divisor** of $b \in \mathcal{D}_t$ if there exists $x \in \mathcal{D}_t$ such that $xa = b \Leftrightarrow \mathcal{D}_t b \subset \mathcal{D}_t a$. An element $a \in \mathcal{D}_t$ is called a **left divisor** of $b \in \mathcal{D}_t$ if there exists $x \in \mathcal{D}_t$ such that $ax = b \Leftrightarrow b\mathcal{D}_t \subset a\mathcal{D}_t$. An element $a \in \mathcal{D}_t$ is called a **total divisor** of $b \in \mathcal{D}_t$ if

$$\mathcal{D}_t b \mathcal{D}_t \subset a\mathcal{D}_t \cap \mathcal{D}_t a.$$

**Lemma B.4** *If $\mathcal{D}_t b \mathcal{D}_t \subset a\mathcal{D}_t$, then $a$ is a total divisor of $b$.*

**Proof** By proposition B.3, $\mathcal{D}_t$ is simple. Thus $\mathcal{D}_t b \mathcal{D}_t = 0$ or $\mathcal{D}_t$. If $\mathcal{D}_t b \mathcal{D}_t = 0$, $b = 0$. Then clearly $a$ is a total divisor of $b$. If $\mathcal{D}_t b \mathcal{D}_t = \mathcal{D}_t$, $b$ is unit. Then $a$ is also unit, that is, $a$ is a total divisor of $b$. □

**Proof of Proposition B.4** It suffices to show that there exist unimodular matrices $U \in \mathcal{D}_t^{g\times g}$ and $V \in \mathcal{D}_t^{q\times q}$ such that

$$URV = \begin{pmatrix} \mathrm{diag}(d_1, \cdots, d_p) & 0 \\ 0 & 0 \end{pmatrix}, \tag{B.4}$$

where $0 \neq d_i \in \mathcal{D}_t$, $p := \mathrm{rank}\, R$, and each $d_i$ is a total divisor of $d_{i+1}$ for $1 \le i \le p-1$. In fact, by lemma B.3, the ring $\mathcal{D}_t$ is simple. Thus the two-sided ideal $\mathcal{D}_t b \mathcal{D}_t$ can only be the zero ideal or $\mathcal{D}_t$ itself. This means that $a$ is a total divisor of $b$ if and only if either $b = 0$ or $a$ is a unit. Hence we conclude that $\deg d_i = 0$, $i = 1, \cdots, p-1$. Furthermore if by elementary operations, $R$ can be brought into the form

$$R' = \begin{pmatrix} d & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & Q & \\ 0 & & & \end{pmatrix}, \tag{B.5}$$

where $d$ is a total divisor of all entries of $Q$, then by applying the same procedure to $Q$, we can show that there exist unimodular matrices $U$ and $V$ satisfying (B.4).
**Case 1:** Suppose that there exist $i$, $j$ such that $R_{ij}$ is a total divisor of all entries of $R$. By a suitable interchange of rows and columns, this element can be brought into the $(1, 1)$ position of the matrix. Therefore without loss of generality, $R_{11}$ is a total divisor of all entries of $R$. This means that $x_i R_{11} = R_{i1}$ and $R_{11} y_j = R_{1j}$. Now perform the following elementary operations:

- For any $i \neq 1$, put $ith$ row minus $x_i$ times $1th$ row.
- For any $j \neq 1$, put $jth$ column minus $1th$ column times $y_j$.

Then we are finished.
**Case 2:** Suppose that there is no $i, j$ such that $R_{ij}$ is a total divisor of all entries of $R$. Let

$$\delta R := \min\{\deg R_{ij} \mid R_{ij} \neq 0\}.$$

Without loss of generality, $\deg R_{11} = \delta R$. We show that we can transform $R$ into $R^{(1)}$ with $\delta R^{(1)} < \delta R$. Then we obtain a strictly decreasing sequence

$$\delta R > \delta R^{(1)} > \delta R^{(2)} > \cdots \geq 0.$$

After finitely many steps, we obtain a matrix which has a unit as an entry, and thus we are in Case 1.
**Case 2.a:** Suppose that $R_{11}$ is not a left divisor of all $R_{1j}$, and that it is not a left divisor of $R_{1k}$. By the Euclidean algorithm, we can write

$$R_{1k} = R_{11} q + r,$$

where $r \neq 0$ and $\deg r < \deg R_{11}$. Perform the elementary operation such that $kth$ column minus $1th$ column times $q$. Then the new matrix $R^{(1)}$ has $r$ in the $(1, k)$ position and thus $\delta R^{(1)} < \delta R$ as desired.
**Case 2.a':** Suppose that $R_{11}$ is not a right divisor of all $R_{i1}$. Proceed analogously as in Case 2.a.
**Case 2.b:** Suppose that $R_{11}$ is a left divisor of all $R_{1j}$, and a right divisor of all $R_{i1}$. Similarly as in Case 1, by elementary operations, we can transform $R$ into the form (B.5). If $a$ is a total divisor of all entries of $Q$, then we are finished. If there exist $i, j$ such that $a$ is not a total divisor of $b := Q_{ij}$, then there exists $c$ such that $a$ is not a left divisor of $cb$. We perform the elementary operation; $1th$ row plus $c$ times $(i+1)th$ row. The new matrix has $cb$ in the $(1, j+1)$ position and therefore we are in Case 2.a. □

Let $C^\infty_{\text{a.e.}}$ denote the set of all functions which are smooth except for a countable set of exception points $\boldsymbol{E}(a) \subset \boldsymbol{R}$ for each $a \in C^\infty_{\text{a.e.}}$, that is, for each $a \in C^\infty_{\text{a.e.}}$ there exists a countable set $\boldsymbol{E}(a) \subset \boldsymbol{R}$ such that $a \in C^\infty(\boldsymbol{R} \backslash \boldsymbol{E}(a), \boldsymbol{R})$. We have the following proposition [118].

**Proposition B.5** *The left $\mathcal{D}_t$ module $C^\infty_{\text{a.e.}}$ is an injective cogenerator.*

Let $R \in \mathcal{D}_t^{g \times q}$. We define the behavior

$$\mathcal{B} := \{w \in (C^\infty_{\text{a.e.}})^q \mid Rw = 0\}.$$

We have the following proposition [118].

**Proposition B.6** *The following are equivalent:*

1. *The behavior $\mathcal{B}$ is autonomous.*
2. *There exists a discrete set $\boldsymbol{E} \subset \boldsymbol{R}$ such that for all open intervals $I \subset \boldsymbol{R} \backslash \boldsymbol{E}$, and all $w \in \mathcal{B}$ which are smooth on $I$, we have*

$$w\big|_J = 0 \quad \text{for all open intervals } J \subset I \Rightarrow w\big|_I = 0.$$

Let

$$URV = \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}$$

be the Jacobson form of $R$, where $D = \mathrm{diag}(1, \cdots, 1, d) \in \mathcal{D}^{p \times p}$, where $0 \neq d \in \mathcal{D}$ and $p := \operatorname{rank} R$. Since $Rw = 0 \Leftrightarrow URw = URVV^{-1}w = 0$, there exists an isomorphism of Abelian groups

$$\begin{aligned} &\mathcal{B} \cong \tilde{\mathcal{B}} := \{\tilde{w} \in (C^{\infty}_{\text{a.e.}})^q \mid \begin{pmatrix} D & 0 \end{pmatrix} \tilde{w} = 0\}, \\ &w \mapsto \tilde{w} := V^{-1}w. \end{aligned} \tag{B.6}$$

Let $\mathcal{M} := \mathcal{D}_t^{1 \times q}/(\mathcal{D}_t^{1 \times g}R)$. Then we have the following lemma.

**Lemma B.5** *There exists an isomorphism of left $\mathcal{D}_t$ module*

$$\mathcal{M} \cong \mathcal{D}_t/(\mathcal{D}_t d) \oplus \mathcal{D}_t^{1 \times (q-p)}, \tag{B.7}$$

*where $0 \neq d \in \mathcal{D}_t$. Moreover, the degree of $d$ is constant.*

**Proof** According to the Jacobson form of $R$, there is an isomorphism of left $\mathcal{D}_t$ modules

$$\mathcal{M} \cong \mathcal{D}_t^{1 \times q}/(\mathcal{D}_t^{1 \times p} \begin{pmatrix} D & 0 \end{pmatrix}).$$

Since $\mathcal{D}_t^{1 \times q}/(\mathcal{D}_t^{1 \times p} \begin{pmatrix} D & 0 \end{pmatrix}) \cong \mathcal{D}_t/(\mathcal{D}_t d) \oplus \mathcal{D}_t^{1 \times (q-p)}$, we have (B.7).

Next, we show that the degree of $d$ in (B.7) is constant. Let

$$\mathcal{M} \cong \mathcal{D}_t/(\mathcal{D}_t d) \oplus \mathcal{D}_t^{1 \times (q-p)} \cong \mathcal{D}_t/(\mathcal{D}_t d') \oplus \mathcal{D}_t^{1 \times (q-p)},$$

where $0 \neq d, d' \in \mathcal{D}_t$. Then we have

$$\mathcal{D}_t/(\mathcal{D}_t d) \cong \mathcal{D}_t/(\mathcal{D}_t d').$$

If the degree of $d$ is not equal to that of $d'$, we do not have the above isomorphism. Therefore the degree of $d$ is constant. $\square$

Lemma B.5 means that although unimodular matrices $U$ and $V$ satisfying (B.3) are not unique, the degree of $d$ in (B.3) is unique. Furthermore, we note that the module $\mathcal{D}_t/(\mathcal{D}_t d)$ is isomorphic to the torsion submodule

$$t\mathcal{M} := \{m \in \mathcal{M} \,|\, 0 \neq \exists e \in \mathcal{D}_t \text{ s.t. } em = 0\}$$

of $\mathcal{M}$. Since by the Malgrange isomorphism,

$$\begin{aligned} \mathrm{Hom}_{\mathcal{D}_t}(\mathcal{D}_t/(\mathcal{D}_t d), C^\infty_{\text{a.e.}}) &\cong \{y \in C^\infty_{\text{a.e.}} \,|\, dy = 0\}, \\ \mathrm{Hom}_{\mathcal{D}_t}(\mathcal{D}_t^{1\times(q-p)}, C^\infty_{\text{a.e.}}) &\cong (C^\infty_{\text{a.e.}})^{q-p}, \end{aligned}$$

the decomposition (B.7) induces an isomorphism of Abelian groups

$$\mathcal{B} \cong \{y \in C^\infty_{\text{a.e.}} \,|\, dy = 0\} \oplus (C^\infty_{\text{a.e.}})^{q-p}. \tag{B.8}$$

**Definition B.6** *The behavior $\mathcal{B}$ is called* **controllable** *if for all $w_1$, $w_2 \in \mathcal{B}$ and for almost all $t_0 \in \boldsymbol{R}$, there exist $w \in \mathcal{B}$, an open interval $t_0 \in I \subset \boldsymbol{R}$, and $t_1 > t_0$ with $t_1 \in I$ such that $w_1$, $w_2$, $w$ are smooth on $I$ and for all $t \in I$*

$$w(t) = \begin{cases} w_1(t), & \text{if} \quad t \leq t_0, \\ w_2(t), & \text{if} \quad t \geq t_1. \end{cases}$$

We have the following proposition [118].

**Proposition B.7** *The behavior $\mathcal{B}$ is controllable if and only if it admits an image representation.*

**Proof** Suppose that $\mathcal{B}$ admits an image representation

$$\mathcal{B} = \{w \in (C^\infty_{\text{a.e.}})^q \,|\, \exists l \in (C^\infty_{\text{a.e.}})^s \text{ s.t. } w = Ll\}.$$

Let $w_1 = Ll_1$, $w_2 = Ll_2 \in \mathcal{B}$ be given and let $t_0$ be in $\boldsymbol{R}\backslash(\boldsymbol{E}(l_1) \cup \boldsymbol{E}(l_2) \cup \boldsymbol{E}(L))$. Then there exists an open interval $t_0 \in I \subset \boldsymbol{R}$ such that $l_1$, $l_2$ and $w_1$, $w_2$ are smooth on $I$. Let $l$ be a smooth function on $I$ with

$$l(t) = \begin{cases} l_1(t) & \text{if} \quad t \leq t_0, \\ l_2(t) & \text{if} \quad t \geq t_1, \end{cases}$$

where $t_1 \in I$ and $t_1 > t_0$. Then $w := Ll$ has the required concatenability property.

For the converse, suppose that $\mathcal{B}$ does not admit an image representation. Then by proposition B.2, the left $\mathcal{D}_t$ module $\mathcal{M}$ is not torsion-free. Hence by lemma B.5, $\mathcal{D}_t/(\mathcal{D}_t d)$ is torsion. Thus proposition B.1 and the relation (B.8) implies that $\mathcal{B}_1 := \{w \in C^\infty_{\text{a.e.}} \,|\, dw = 0\}$ is autonomous. Let $w_1$ be the zero solution, and let $w_2$ be a non-zero solution. Then there exists an open interval $I_0 \subset \boldsymbol{R}\backslash\boldsymbol{E}(d)$ on which $w_2$ is smooth and does not vanish. Let $t_0 \in I_0$. Suppose that $w$ was a connecting trajectory. Then $w$ is smooth on some open neighborhood $I \subset I_0$ of $t_0$. On the other hand, by proposition B.6, $w(t) = w_1(t) = 0$ for all $t \in I$ with $t \leq t_0$ implies that $w(t) = 0$ for all $t \in I$. This contradicts $w(t) = w_2(t) \neq 0$ for all $t \in I$ with $t \geq t_1 > t_0$. $\square$

By propositions B.2 and B.7, we have the following proposition [118].

**Proposition B.8** *The behavior $\mathcal{B}$ is controllable if and only if there exist unimodular matrices $U \in \mathcal{D}_t^{g\times g}$ and $V \in \mathcal{D}_t^{q\times q}$ such that*

$$URV = \begin{pmatrix} I_p & 0 \\ 0 & 0 \end{pmatrix},$$

*where $p := \operatorname{rank} R$.*

**Proof** First note that by proposition B.4, there exist unimodular matrices $U \in \mathcal{D}_t^{g\times g}$ and $V \in \mathcal{D}_t^{q\times q}$ satisfying (B.3). By propositions B.2 and B.7, the behavior $\mathcal{B}$ is controllable if and only if $\mathcal{M} := \mathcal{D}_t^{1\times q}/(\mathcal{D}_t^{1\times g}R)$ is torsion-free. Hence lemma B.5 implies that $\mathcal{B}$ is controllable if and only if $\mathcal{D}_t/(\mathcal{D}_t d) = \{0\}$. Since $\mathcal{D}_t/(\mathcal{D}_t d) = \{0\} \Leftrightarrow d \in \mathcal{M}_t\backslash\{0\}$, we have the conclusion. $\square$

# Appendix C

# Pseudo-linear algebra

For the convenience of readers, we summarize some results of pseudo-linear algebra [8]. Let $K$ be a field and $\sigma : K \to K$ an injective endomorphism. A map $\delta : K \to K$ is called a **pseudo-derivation** if it satisfies

$$\begin{cases} \delta(a+b) = \delta(a) + \delta(b), \\ \delta(ab) = \sigma(a)\delta(b) + \delta(a)b. \end{cases}$$

If $\sigma(a) = a$ for any $a \in K$, the pair $(K, \delta)$ is called a **differential field**.

The left skew polynomial ring given by $\sigma$ and $\delta$ is the ring $(K[s]; \sigma, \delta)$ of polynomials in $s$ over $K$ with the usual addition and the non-commutative multiplication given by the commutative rule

$$sa = \sigma(a)s + \delta(a)$$

for any $a \in K$.

Let $V$ be a vector space over $K$. A map $\theta : V \to V$ is called **pseudo-linear** if

$$\begin{cases} \theta(u+v) = \theta(u) + \theta(v), \\ \theta(au) = \sigma(a)\theta(u) + \delta(a)u \end{cases}$$

for any $a \in K$, and $u, v \in V$.

Skew polynomials can act on a vector space. Let

$$\begin{aligned} &\theta^k u := \theta(\theta^{k-1}(u)) \quad \text{for any} \quad k \geq 1, \\ &\theta^0 u := u. \end{aligned}$$

Any pseudo-linear map $\theta : V \to V$ induces the action $* : (K[s]; \sigma, \delta) \times V \to V$ defined by

$$\left( \sum_{i=0}^{n} a_i s^i \right) * u = \sum_{i=0}^{n} a_i \theta^i(u)$$

for any $u \in V$.

# Appendix D

# Analytic function and meromorphic function

For the convenience of readers, we give the definitions of analytic function and meromorphic function based on reference [16]. First, we define analytic function.

**Definition D.1** *A function* $f : \boldsymbol{R}^n \to \boldsymbol{R}$ *is called* **analytic** *if it coincides with its Taylor expansion*

$$f(x_1, \cdots, x_n) = \sum_{i_1=0}^{\infty} \cdots \sum_{i_n=0}^{\infty} a_{i_1,\cdots,i_n}(x_1 - x_1^0)^{i_1} \cdots (x_n - x_n^0)^{i_n}$$

*in the neighborhood of any* $x^0 \in \boldsymbol{R}^n$*.*

Let $\mathcal{A}$ be the set of analytic functions from $\boldsymbol{R}^n$ to $\boldsymbol{R}$. The following proposition has been known as the identity theorem.

**Proposition D.1** *Let* $f \in \mathcal{A}$*. Then*

1. $f \equiv 0$ *on* $\boldsymbol{R}^n$*, or*

2. *the set of zeros of* $f$ *is measure zero.*

Proposition D.1 implies the following lemma.

**Lemma D.2** *The set* $\mathcal{A}$ *is a domain (see appendix A).*

**Proof** Let $f$, $g \in \mathcal{A}$, $f, g \neq 0$ on $\boldsymbol{R}^n$, and let

$$S_f := \{x \in \boldsymbol{R}^n \mid f(x) = 0\},$$
$$S_g := \{x \in \boldsymbol{R}^n \mid g(x) = 0\}.$$

If $S_f$ or $S_g$ are not measure zeros, by proposition D.1, we must conclude $f = 0$ or $g = 0$ on $\boldsymbol{R}^n$. This is a contradiction. Hence $S_f$ and $S_g$ are both measure zeros. Thus $S_f \cup S_g$ is also measure zero. Therefore $f \cdot g \neq 0$. $\square$

The above lemma yields $\mathcal{A} \subsetneq C^\infty(\boldsymbol{R}^n, \boldsymbol{R})$. In fact, let

$$f_1(x) = \begin{cases} \exp(-\frac{1}{x^2}), & \text{if} \quad x < 0, \\ 0, & \text{if} \quad x \geq 0, \end{cases}$$

$$f_2(x) = \begin{cases} 0, & \text{if} \quad x \leq 0, \\ \exp(-\frac{1}{x^2}), & \text{if} \quad x > 0. \end{cases}$$

Then $0 \neq f_1,\ f_2 \in C^\infty(\boldsymbol{R}, \boldsymbol{R})$ and $f_1 \cdot f_2 = 0$. Since $\mathcal{A}$ is a commutative domain, by proposition A.3, we can construct the **quotient field** of $\mathcal{A}$.

**Definition D.3** *The elements of the quotient field of $\mathcal{A}$ are called* **meromorphic** *functions.*

We note that

1. if we substitute an analytic function into a meromorphic function, the resulting function is a meromorphic function.

2. if we substitute a meromorphic function into a meromorphic function, the resulting function may not be a meromorphic function.

For example, if we substitute a meromorphic function $x = \frac{1}{t}$ into a meromorphic function $\sin x$, we have $\sin \frac{1}{t}$. However, $\sin \frac{1}{t}$ is not meromorphic.

# Appendix E

# Geometric interpretation of differential flatness

For the convenience of readers, we summarize a geometric interpretation of differential flatness. We refer to [22, 23, 65, 72].

## E.1 Control systems as infinite dimensional vector fields

Let us consider

$$\dot{x} = f(x, u), \tag{E.1}$$

where $x \in \boldsymbol{R}^n$ and $u \in \boldsymbol{R}^m$ denote state and input variables, respectively, and rank $\frac{\partial f}{\partial u} = m$. It is possible to associate to (E.1) an extended vector field having the same solutions in the following manner: We start by considering the infinite mapping

$$t \mapsto \xi(t) = (x(t), u(t), \dot{u}(t), \cdots) \in \boldsymbol{R}^n \times \boldsymbol{R}^m \times \boldsymbol{R}^m_\infty, \tag{E.2}$$

where $\boldsymbol{R}^m_\infty$ is an infinite dimensional vector space whose coordinates are of the form $(\dot{u}, \ddot{u}, \cdots)$ with $u^{(i)} \in \boldsymbol{R}^m$, $i \geq 1$. The space $\boldsymbol{R}^m_\infty$ is the projective limit of $\boldsymbol{R}^m_k$, $k \geq 1$ with coordinates $(\dot{u}, \ddot{u}, \cdots, u^{(k)})$. It is convenient to use the symbol $\boldsymbol{R}^m_0$ for $k = 0$ to define $\boldsymbol{R}^n \times \boldsymbol{R}^m \times \boldsymbol{R}^m_0 := \boldsymbol{R}^n \times \boldsymbol{R}^m$. The projections $\pi_k$, $k \geq 1$ from $\boldsymbol{R}^m_\infty$ to $\boldsymbol{R}^m_k$ is given by

$$\pi_k(\dot{u}, \ddot{u}, \cdots) := (\dot{u}, \ddot{u}, \cdots, u^{(k)}).$$

The topology of $\boldsymbol{R}^m_\infty$ is the product topology, that is, an open set of $\boldsymbol{R}^m_\infty$ is of the form $\pi_k^{-1}(O)$ with $O$ an open subset of $\boldsymbol{R}^m_k$. A function on $\boldsymbol{R}^m_\infty$ is called smooth if it depends on a finite but arbitrary number of variables and is smooth in the usual sense.

Given a smooth solution of (E.1), the mapping (E.2) satisfies

$$\dot{\xi}(t) = (f(x(t), u(t)), \dot{u}(t), \ddot{u}(t), \cdots),$$

which implies that $\xi(t)$ can be viewed as a trajectory of the infinite dimensional vector field

$$(x, u, \dot{u}, \cdots) \mapsto F(x, u, \dot{u}, \cdots) := (f(x, u), \dot{u}, \ddot{u}, \cdots)$$

on $\boldsymbol{R}^n \times \boldsymbol{R}^m \times \boldsymbol{R}^m_\infty$. Conversely, any mapping

$$t \mapsto \xi(t) = (x(t), u(t), \dot{u}(t), \cdots)$$

with $\dot{x}(t) = f(x(t), u(t))$ corresponds to a solution of (E.1). Therefore, the vector field $F$ is the extended vector field on $\boldsymbol{R}^n \times \boldsymbol{R}^m \times \boldsymbol{R}^m_\infty$, which we wanted to find.

We are now in a position to give a formal definition of an infinite dimensional system.

**Definition E.1** *A system is a pair* $(\boldsymbol{R}^n \times \boldsymbol{R}^m \times \boldsymbol{R}^m_\infty, F)$*, where* $F$ *is a smooth vector field on* $\boldsymbol{R}^n \times \boldsymbol{R}^m \times \boldsymbol{R}^m_\infty$*.*

We note that the extended vector field $F(x, u, \dot{u}, \cdots) := (f(x, u), \dot{u}, \ddot{u}, \cdots)$ can be identified with the original dynamics $\dot{x} = f(x, u)$.

## E.2 Lie-Bäcklund equivalence of systems

In this section, we define an equivalence relation among systems. Let us consider two systems $(\boldsymbol{R}^n \times \boldsymbol{R}^m \times \boldsymbol{R}^m_\infty, F)$ and $(\boldsymbol{R}^{\tilde{n}} \times \boldsymbol{R}^{\tilde{m}} \times \boldsymbol{R}^{\tilde{m}}_\infty, G)$, and a smooth mapping $\Psi : \boldsymbol{R}^n \times \boldsymbol{R}^m \times \boldsymbol{R}^m_\infty \to \boldsymbol{R}^{\tilde{n}} \times \boldsymbol{R}^{\tilde{m}} \times \boldsymbol{R}^{\tilde{m}}_\infty$. If $t \mapsto \xi(t)$ is a trajectory of $(\boldsymbol{R}^n \times \boldsymbol{R}^m \times \boldsymbol{R}^m_\infty, F)$, the composed mapping $t \mapsto \xi(t) = \Psi(\xi(t))$ satisfies, by the chain rule,

$$\dot{\zeta} = \frac{\partial \Psi}{\partial \xi}(\xi(t))\dot{\xi}(t) = \frac{\partial \Psi}{\partial \xi}(\xi(t))F(\xi(t)).$$

Now, if the vector fields $F$ and $G$ are $\Psi$-related, that is, for any $\xi \in \boldsymbol{R}^n \times \boldsymbol{R}^m \times \boldsymbol{R}^m_\infty$,

$$G(\Psi(\xi)) = \frac{\partial \Psi}{\partial \xi}(\xi)F(\xi),$$

then

$$\dot{\xi}(t) = G(\Psi(\xi(t))) = G(\zeta(t)),$$

which means that $t \mapsto \zeta(t) = \Psi(\xi(t))$ is a trajectory of $(\boldsymbol{R}^{\tilde{n}} \times \boldsymbol{R}^{\tilde{m}} \times \boldsymbol{R}^{\tilde{m}}_\infty, G)$. If moreover $\Psi$ has a smooth inverse $\Phi$, then $F$ and $G$ are also $\Phi$-related, and there is a one-to-one correspondence between the trajectories of the two systems. We call such an invertible mapping $\Psi$ relating $F$ and $G$ an **endogenous transformation**.

**Definition E.2** *Two systems* $(\boldsymbol{R}^n \times \boldsymbol{R}^m \times \boldsymbol{R}^m_\infty, F)$ *and* $(\boldsymbol{R}^{\tilde{n}} \times \boldsymbol{R}^{\tilde{m}} \times \boldsymbol{R}^{\tilde{m}}_\infty, G)$ *are called* **Lie-Bäcklund equivalent** *at* $(p, q) \in (\boldsymbol{R}^n \times \boldsymbol{R}^m \times \boldsymbol{R}^m_\infty) \times (\boldsymbol{R}^{\tilde{n}} \times \boldsymbol{R}^{\tilde{m}} \times \boldsymbol{R}^{\tilde{m}}_\infty)$ *if there exists an endogenous transformation from a neighborhood of* $p$ *to a neighborhood of* $q$*. Two systems* $(\boldsymbol{R}^n \times \boldsymbol{R}^m \times \boldsymbol{R}^m_\infty, F)$ *and* $(\boldsymbol{R}^{\tilde{n}} \times \boldsymbol{R}^{\tilde{m}} \times \boldsymbol{R}^{\tilde{m}}_\infty, G)$ *are called* **Lie-Bäcklund equivalent** *if they are Lie-Bäcklund equivalent at every pair of points* $(p, q) \in (\boldsymbol{R}^n \times \boldsymbol{R}^m \times \boldsymbol{R}^m_\infty) \times (\boldsymbol{R}^{\tilde{n}} \times \boldsymbol{R}^{\tilde{m}} \times \boldsymbol{R}^{\tilde{m}}_\infty)$.

We note that if two systems are Lie-Bäcklund equivalent, there is an invertible transformation exchanging their trajectories.

An important property of endogenous transformations is that they preserve the number of input variables [22, 23, 65, 72].

**Proposition E.1** *If two systems* $(\boldsymbol{R}^n \times \boldsymbol{R}^m \times \boldsymbol{R}^m_\infty, F)$ *and* $(\boldsymbol{R}^{\tilde{n}} \times \boldsymbol{R}^{\tilde{m}} \times \boldsymbol{R}^{\tilde{m}}_\infty, G)$ *are Lie-Bäcklund equivalent, then they have the same number of input variables, that is,* $m = \tilde{m}$.

## E.3 Differential flatness

In this section, we give a geometrical definition of differential flatness.

**Definition E.3** *A system* $(\boldsymbol{R}^m \times \boldsymbol{R}^m_\infty, F_m)$ *is called* **trivial** *if the vector field* $F_m$ *can be expressed as*

$$F_m := \sum_{1 \le i \le m,\, 0 \le j} v_i^{(j+1)} \frac{\partial}{\partial v_i^{(j)}}. \tag{E.3}$$

**Definition E.4** *A system* $(\boldsymbol{R}^n \times \boldsymbol{R}^m \times \boldsymbol{R}^m_\infty, F)$ *is called* **differentially flat** *if it is Lie-Bäcklund equivalent to a trivial system, where* $(v_1, \cdots, v_m)$ *of* (E.3) *is called a* **flat output**.

We say that if a system $(\boldsymbol{R}^n \times \boldsymbol{R}^m \times \boldsymbol{R}^m_\infty, F)$ is differentially flat, the corresponding system (E.1) is differentially flat.

Since we have the following proposition [23], we have adopted definition 2.10 as the definition of differential flatness.

**Proposition E.2** *System* (E.1) *is differentially flat if and only if there exist smooth mappings* $\phi_1 : \boldsymbol{R}^m \times \boldsymbol{R}^m \times \cdots \to \boldsymbol{R}^n$, $\phi_2 : \boldsymbol{R}^m \times \boldsymbol{R}^m \times \cdots \to \boldsymbol{R}^m$, *and* $\psi : \boldsymbol{R}^n \times (\boldsymbol{R}^m \times \cdots) \to \boldsymbol{R}^m$ *depending only on a finite number of variables, respectively, such that*

$$v := \psi(x, u, \dot{u}, \cdots) \Rightarrow \begin{pmatrix} x \\ u \end{pmatrix} = \begin{pmatrix} \phi_1(v, \dot{v}, \ddot{v}, \cdots) \\ \phi_2(v, \dot{v}, \ddot{v}, \cdots) \end{pmatrix}.$$

# Appendix F

# Trajectory tracking control based on exact feedback linearization

For the convenience of readers, we summarize some results of trajectory tracking control based on exact feedback linearization [36, 82]. For simplicity, let us consider an affine single input single output (SISO) nonlinear control system

$$\dot{x} = f(x) + g(x)u, \tag{F.1}$$

$$y = h(x), \tag{F.2}$$

where $x \in \boldsymbol{R}^n$, $u \in \boldsymbol{R}$, and $y \in \boldsymbol{R}$ are state, input, and output variables, respectively, and $f \in C^\infty(\boldsymbol{R}^n, \boldsymbol{R}^n)$, $g \in C^\infty(\boldsymbol{R}^n, \boldsymbol{R}^n)$, and $h \in C^\infty(\boldsymbol{R}^n, \boldsymbol{R})$.

To design a controller for a trajectory tracking control of system (F.1)-(F.2), let us transform system (F.1)-(F.2) into a normal form. To this end, first, we define the concept of relative degree [36, 82]. Let $\lambda \in C^\infty(\boldsymbol{R}^n, \boldsymbol{R})$ and $F \in C^\infty(\boldsymbol{R}^n, \boldsymbol{R}^n)$. Then the **Lie derivative** of $\lambda$ along $F$ is defined as

$$L_F\lambda(x) := \sum_{i=1}^{n} \frac{\partial \lambda}{\partial x_i}(x) F_i(x).$$

**Definition F.1** *System* (F.1)-(F.2) *is said to have* **relative degree** $r$ *at a point* $x^0$ *if*

1. $L_g L_f^k h(x) = 0$ *for all* $x$ *in a neighborhood of* $x^0$ *and all* $k < r - 1$.
2. $L_g L_f^{r-1} h(x^0) \neq 0$.

By using the concept of relative degree, we have the following proposition [36].

**Proposition F.1** *Let* $r$ *be the relative degree at* $x^0$ *of system* (F.1)-(F.2). *Then*

$$dh(x^0),\ dL_f h(x^0), \cdots, dL_f^{r-1} h(x^0)$$

*are linearly independent.*

Proposition F.1 shows that $r \leq n$ and the $r$ functions

$$h(x),\ L_f h(x), \cdots,\ L_f^{r-1} h(x)$$

quality as a partial set of new coordinate functions around the point $x^0$. Now, let

$$\begin{cases} \phi_1(x) := h(x), \\ \phi_2(x) := L_f h(x), \\ \vdots \\ \phi_r(x) := L_f^{r-1} h(x), \end{cases} \tag{F.3}$$

and let $\phi_{r+1}(x), \cdots, \phi_n(x)$ such that the Jacobian matrix of

$$\Phi(x) := \begin{pmatrix} \phi_1(x) \\ \vdots \\ \phi_n(x) \end{pmatrix}$$

is nonsingular at $x^0$. Then $z_i = \phi_i(x)$, $1 \leq i \leq n$ are new coordinates. By a direct calculation, we have

$$\begin{aligned}
\dot{z}_1 &= \frac{\partial \phi_1}{\partial x}\dot{x} = \frac{\partial h}{\partial x}(f(x) + g(x)u) = L_f h(x) + L_g h(x) = L_f h(x) = z_2, \\
&\vdots \\
\dot{z}_{r-1} &= \frac{\partial \phi_{r-1}}{\partial x}\dot{x} = \frac{\partial (L_f^{r-2} h)}{\partial x}(f(x) + g(x)u) = L_f^{r-1} h(x) = z_r, \\
\dot{z}_r &= \frac{\partial \phi_r}{\partial x}\dot{x} = \frac{\partial (L_f^{r-1} h)}{\partial x}(f(x) + g(x)u) = L_f^r h(x) + L_g L_f^{r-1} h(x) u, \\
\dot{z}_{r+1} &= \frac{\partial \phi_{r+1}}{\partial x}\dot{x} = \frac{\partial \phi_{r+1}}{\partial x}(f(x) + g(x)u), \\
&\vdots \\
\dot{z}_n &= \frac{\partial \phi_n}{\partial x}\dot{x} = \frac{\partial \phi_n}{\partial x}(f(x) + g(x)u).
\end{aligned}$$

Hence if we set

$$\begin{aligned}
a(z) &:= L_g L_f^{r-1} h(\Phi^{-1}(z)), \\
b(z) &:= L_f^r h(\Phi^{-1}(z)), \\
q_i(z) &:= \frac{\partial \phi_i}{\partial x}(\Phi^{-1}(z)) f(\Phi^{-1}(z)), \\
p_i(z) &:= \frac{\partial \phi_i}{\partial x}(\Phi^{-1}(z)) g(\Phi^{-1}(z)),
\end{aligned}$$

where $r+1 \le i \le n$, system (F.1)-(F.2) can be transformed into

$$\begin{cases} \dot{z}_1 = z_2, \\ \quad\vdots \\ \dot{z}_{r-1} = z_r, \\ \dot{z}_r = b(z) + a(z)u, \\ \dot{z}_{r+1} = q_{r+1}(z) + p_{r+1}(z)u, \\ \quad\vdots \\ \dot{z}_n = q_n(z) + p_n(z)u, \\ y = z_1. \end{cases} \tag{F.4}$$

Furthermore, we have the following proposition (see proposition 4.1.3 in [36]).

**Proposition F.2** *Suppose that system* (F.1)-(F.2) *has relative degree $r$ at $x^0$, and that $\phi_1, \cdots, \phi_r$ are defined as* (F.3)*. Then it is possible to choose $\phi_{r+1}(x)$, $\cdots$, $\phi_n(x)$ such that*

$$\begin{aligned} L_g\phi_i(x) &= 0, \quad r+1 \le i \le n, \quad \text{for all } x \text{ around } x^0, \qquad \text{(F.5)} \\ \det \frac{\partial \Phi}{\partial x}(x^0) &\neq 0. \end{aligned}$$

If we choose $\phi_{r+1}(x)$, $\cdots$, $\phi_n(x)$ satisfying (F.5),

$$\dot{z}_i = \frac{\partial \phi_i}{\partial x}\dot{x} = L_f\phi_i(x(t)) + L_g\phi(x(t))u(t) = L_f\phi(x(t)), \quad r+1 \le i \le n.$$

Therefore around $x^0$, system (F.4) can be transformed into

$$\begin{cases} \dot{z}_1 = z_2, \\ \quad\vdots \\ \dot{z}_{r-1} = z_r, \\ \dot{z}_r = b(z) + a(z)u, \\ \dot{z}_{r+1} = q_{r+1}(z), \\ \quad\vdots \\ \dot{z}_n = q_n(z), \\ y = z_1. \end{cases}, \tag{F.6}$$

where $q_i(z) := L_f\phi_i(\Phi^{-1}(z))$, $r+1 \le i \le n$.

**Remark F.1** *In general, it is difficult to construct $n-r$ functions $\phi_{r+1}(x)$, $\cdots$, $\phi_n(x)$ satisfying $L_g\phi_i(x) = 0$ because we have to solve $n-r$ partial differential equations.* ■

Now, we define $\xi := (z_1, \cdots, z_r)$ and $\eta := (z_{r+1}, \cdots, z_n)$. Then system (F.6) can be expressed as

$$\begin{cases} \dot{z}_1 = z_2, \\ \quad \vdots \\ \dot{z}_{r-1} = z_r, \\ \dot{z}_r = b(\xi, \eta) + a(\xi, \eta)u, \\ \dot{\eta} = q(\xi, \eta), \\ y = z_1. \end{cases} \tag{F.7}$$

If we apply

$$u = \frac{1}{a(\xi, \eta)} \left( -b(\xi, \eta) + y_R^{(r)} - \sum_{i=1}^{r} c_{i-1}(z_i - y_R^{(r)}) \right), \tag{F.8}$$

we have

$$\dot{z}_r = y^{(r)} = y_R^{(r)} - c_{r-1}e^{(r-1)} - \cdots - c_1 e^{(1)} - c_0 e, \tag{F.9}$$

where $e(t) := y(t) - y_R(t)$, and get

$$e^{(r)} + c_{r-1}e^{(r-1)} + \cdots + c_1 e^{(1)} + c_0 e = 0.$$

**Remark F.2** *The control law* (F.8) *can be regarded as a generalized two-degree-of-freedom control because*

$$u = \underbrace{\left( \frac{-b(\xi, \eta)}{a(\xi, \eta)} + \frac{y_R^{(r)}}{a(\xi, \eta)} \right)}_{\text{feedforward}} + \underbrace{\left( -\frac{1}{a(\xi, \eta)} \sum_{i=1}^{r} c_{i-1}(z_i - y_R^{(r)}) \right)}_{\text{feedback}}$$

*Note that the feedforward controller also uses information of the current state.* ■

We have a sufficient condition for the boundedness of $z_i(t)$, $1 \leq i \leq r$ and $\eta(t)$ (see proposition 4.5.1 in [36]).

**Proposition F.3** *Suppose that $y_R(t)$, $y_R^{(1)}(t)$, $\cdots$, $y_R^{(r-1)}(t)$ are defined for all $t \geq 0$ and bounded. Moreover, suppose that $\eta_R(t)$ denotes the solution of*

$$\eta = q(\xi_R(t), \eta)$$

*satisfying $\eta_R(0) = 0$ and is defined for all $t \geq 0$, bounded and uniformly asymptotically stable. Furthermore assume that the roots of the polynomial*

$$s^r + c_{r-1}s^{r-1} + \cdots + c_1 s + c_0 = 0$$

*all have negative real part. Then if for sufficiently small $a > 0$,*

$$|z_i(t^0) - y_R^{(i-1)}(t^0)| < a, \ 1 \le i \le r, \quad ||\eta(t^0) - \eta_R(t_0)|| < a,$$

*for all $\epsilon > 0$, there exists $\delta > 0$ such that*

$$\begin{aligned} |z_i(t^0) - y_R^{(i-1)}(t^0)| < \delta &\Rightarrow |z_i(t) - y_R^{(i-1)}(t)| < \epsilon, \\ |\eta(t^0) - \eta_R(t^0)| < \delta &\Rightarrow |\eta(t) - \eta_R(t)| < \epsilon \end{aligned}$$

*for all $t \ge t^0 \ge 0$.*

**We note that the above discussions can be extended to an affine multi input multi output (MIMO) nonlinear control system** [36, 82].

## F.1 Zero dynamics

Let us consider the output constraint $y(t) = 0$ for all $t$. Since $y(t) = z_1(t)$, the constraint $y(t) = 0$ for all $t$ yields

$$\xi(t) = 0 \quad \text{for all } t.$$

Hence then $\eta(t)$ must satisfy

$$\dot{\eta}(t) = q(0, \eta(t)). \tag{F.10}$$

The dynamics (F.10) is called the **zero dynamics** which describes internal behavior of system (F.1)-(F.2) when input and initial conditions have been chosen in such a way that the output is constrained to identically zero.

We can extend the output constraint $y(t) = 0$ to $y(t) = y_R(t)$ which is any function. In fact, $y(t) = y_R(t)$ implies

$$z_i(t) = y_R^{(i-1)}(t), \quad 1 \le i \le r.$$

Putting $\xi_R(t) = (y_R(t), \dot{y}_R(t), \cdots, y_R^{(r-1)}(t))$, the input $u(t)$ has to satisfy

$$u(t) = \frac{y_R^{(r)}(t) - b(\xi_R(t), \eta(t))}{a(\xi_R(t), \eta(t))}, \tag{F.11}$$

where $\eta(t)$ is a solution of the differential equation

$$\eta(t) = q(\xi_R(t), \eta(t)). \tag{F.12}$$

Eqs. (F.11)-(F.12) is called a **left inverse** [36, 82] of system (F.1)-(F.2) because Eqs. (F.11)-(F.12) express a system with input $\xi_R(t)$, output $u(t)$, and state $\eta(t)$.

The following proposition shows one theoretical limit to the tracking performance that can be obtained in systems with zero dynamics [26].

**Proposition F.4** *Suppose that system* (F.1)-(F.2)

1. *is analytic.*
2. *has a zero dynamics.*
3. *has left invertible.*
4. *has a controllable linearization at* $(x, u) = (0, 0)$.

*Let* $Y(\epsilon, N) := \{y(t) \mid ||y(t)|| \leq \epsilon, \cdots, ||y^{(N)}(t)|| \leq \epsilon, \ \forall t\}$. *Then if there exist an control input* $u$ *and all initial condition in some open set such that* $||y(t) - y_R(t)|| \to 0$, $y_R(t) \in Y(\epsilon, N)$ *for any* $N, \epsilon > 0$, *then system* (F.1)-(F.2) *has asymptotically stable zero dynamics.*

## F.2 Chained form

Exact feedback linearization method reduces nonlinear terms in a given differential equations and transforms into a simpler equation. For a nonholonomic system, by eliminating nonlinear terms, we can obtain a simpler form called **chained form** [78, 79]. However, such a method possesses difficult points. For example, let us consider

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix} u_1 + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u_2. \tag{F.13}$$

If we apply

$$u_1 = \frac{v_1}{\cos\theta}, \tag{F.14}$$

system (F.13) is transformed into

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} v_1 \\ v_1 \tan\theta \\ u_2 \end{pmatrix}, \tag{F.15}$$

where $v_1$ is a new input variable. Furthermore, let $z := \tan\theta$ and $u_2 = \cos^2(\theta) v_2$. Then system (F.15) is transformed into

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} v_1 \\ z v_1 \\ v_2 \end{pmatrix}. \tag{F.16}$$

The form (F.16) is called **chained form** of original system (F.13) [78, 79].

However, (F.14) yields a singularity at $\theta = \frac{\pi}{2} \pm n\pi$, $n \in \boldsymbol{Z}$. Therefore, if we want to track reference trajectories such as (4.35) and (4.55), we should not transform (F.13) into the chained form (F.16).

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# Published Papers by the Author

## Chapter 2

- K. Sato and T. Iwai, "Configuration flatness of Lagrangian control systems with fewer controls than the degrees of freedom ," *Systems & Control Letters*, vol. 61, no. 4, pp. 334–342, 2012.

- K. Sato, "Algebraic controllability of nonlinear mechanical control systems," *SICE Journal of Control, Measurement, and System Integration* (conditionally accepted).

## Chapter 3

- K. Sato, "Algebraic observability of nonlinear differential algebraic systems with geometric index one," in *Proceedings of the 52nd IEEE Conference on Decision and Control*, pp. 2582–2587, 2013.

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## Chapter 4

- K. Sato, "Flatness-based tracking control of nonlinear differential algebraic systems with geometric index one," in *Proceedings of the 52nd IEEE Conference on Decision and Control*, pp. 7443–7448, 2013.

## Related paper

- K. Sato, "Differential flatness of affine nonlinear control systems," in *SICE Annual Conference 2012*, pp. 892–897, 2012.

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